Adjustable Automorphisms Appearing in Composition of Subfactors

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1 Composition of subfactors

Let $\mathbf{N} \subset \mathbf{M}$ be inclusion of factors, and let $\Gamma(\mathbf{M}, \mathbf{N})$ be the group of adjustable automorphisms of $\mathbf{M}$ appearing in the dual principal graph of $\mathbf{N} \subset \mathbf{M}$, after suitable inner pertubation. Those adjustable automorphisms, indeed, are the centrally trivial automorphisms in the sense of Kawahigashi ([4]). Those automorphisms play important role in the recent analysis on subfactors.

It is known that those automorphisms turn to be related to 1-dim bimodules appearing in the dual principapl graph ([6]). In particular, if the depth is two, the group $\Gamma(\mathbf{M}, \mathbf{N})$ is itself Galois group $Gal(\mathbf{M}, \mathbf{N})$, i.e. the instrinsic group of the related Hopf algebras.

In the present article, we analyse the group $\Gamma(\mathbf{M}, \mathbf{N})$ by analysing 1-dim bimodules appearing in the principal graph (or its dual principal graph) of the composition of subfactors.

Let $\mathbf{P}$ be the type II_1 factor, and α and β be outer automorphisms of $\mathbf{P}$ with period 2. We assume that the outer period of $\alpha\beta$ is $2n$. Then we have $(\alpha\beta)^{2n} = Adu$ for a unitary $u \in \mathbf{P}$, and can obtain Connes obstruction ω satisfying $\omega^{2n} = 1$ with $\alpha\beta(u) = \omega u$.

Let $K \cong \{id, \beta\} \cong Z_2$ and $H \cong \{id, \alpha\} \cong Z_2$ and $H \cap K = \{id\}$. We construct a pair of factors

$$\mathbf{N} = \mathbf{P}^K \subset \mathbf{P} \times_\alpha H = \mathbf{M}.$$

Then $\mathbf{N} \subset \mathbf{M}$ is an irreducible subfactor with Jones index 4. It is known that the depth of $\mathbf{N} \subset \mathbf{M}$ depends on the group generated by H and K.

In our cases, we can write down all the relevant irreducible bimodules in the principal graph and their relative tensor products (i.e. fusion rules) with help of the twisted cocycles. We begin with some observation for our aim.

Note that the corresponding **N**-**M** bimodule X is given by

$$X =_N L^2(\mathbf{M})_M = A \otimes_P B$$

when $A =_N L^2(\mathbf{P})_P$, **N**-**P** bimodule and $B =_P L^2(\mathbf{M})_M$, **P**-**M** bimodule. If there is no confusion, we denote by $\hat{\alpha}$ the dual action of an action α, and $\alpha = {}_P{}^{\alpha}L^2(\mathbf{P})_P$, **P**-**P** bimodule (with statistical dimension 1) equipped with the actions $x \cdot \xi \cdot y = \pi(x) J_P y^* J_P \xi$ for $\xi \in L^2(\mathbf{P})$ and $x, y \in \mathbf{P}$. Then we see that $A^* \otimes A = id \oplus \beta$, $A \otimes A^* = id \oplus \hat{\beta}$, $B \otimes B^* = id \oplus \alpha$, and $B^* \otimes B = id \oplus \hat{\alpha}$.

Proposition 1.1 *We have the following*

1. $X \otimes X^* = id \oplus \hat{\beta} \oplus A\alpha A^*$, *as* **N**-**N** *bimodules,*
2. $X^* \otimes X = id \oplus \hat{\alpha} \oplus B^* \beta B$, *as* **M**-**M** *bimodules.*

Since we are interested in **M**-**M** bomodules with statistical dimension 1 which correspond to the automorphisms of **M**, we deal with $(X^* \otimes X)^n$.

2 Adjustable automorphisms in the dual principal graph

The third level relative commutant $X \otimes X^* \otimes X$ (or, $X^* \otimes X \otimes X^*$) depends on the outer period of the automorphism $\alpha\beta$. From now on, we assume that the outer period of $\alpha\beta$ is 2. Then $(\alpha\beta)^2 = Adu$ and $\alpha\beta(u) = \omega u$ with Connes obstruction ω (i.e. $\omega^2 = 1$). Moreover, we may assume $u\alpha(u) = I$ and $u\beta(u) = \omega I$, after a suitable scala multiplication.

Lemma 2.1 $\mathbf{N} \subset \mathbf{M}$ *has depth two.*

Proof. Since we have

$$dimEnd(B^*\beta B) = dimHom(B^*\beta B, B^*\beta B) = dimHom(BB^*\beta, \beta BB^*)$$
$$= dimHom(\beta \oplus \beta\alpha, \beta \oplus \beta\alpha) = 2,$$

it is clear that $B^*\beta B = \pi_1 \oplus \pi_2$ for some 1-dim **P**-**P** bimodules π_i. Thus, $X^* \otimes X = id \oplus \hat{\alpha} \oplus \pi_1 \oplus \pi_2$, the commutative algebra of dimension 4. We also have

$$dimEnd(X^* \otimes X \otimes X^*) = dimHom(id \oplus \hat{\alpha} \oplus \pi_1 \oplus \pi_2, id \oplus \hat{\alpha} \oplus \pi_1 \oplus \pi_2)$$
$$= dimHom(2 \oplus 2\alpha \oplus 2\beta \oplus 2\alpha\beta, 2 \oplus 2\alpha \oplus 2\beta \oplus 2\alpha\beta) = 16,$$

due to Frobenius Reciprocity. It now obvious that $\mathbf{N} \subset \mathbf{M}$ is of depth two. □

Thanks to Lemma 2.1, we can conclude that there exists an outer action of a group G of order 4 such that $\mathbf{M} = \mathbf{N} \times G$. It is clear that the principal graph (and also the dual principal graph) is the 4-star graph $T(2,2,2,2)$ or $D_4^{(1)}$. Therefore we can conclude that there are four 1-dim **M**-**M** bimodules in the principal graph (and its dual principal graph). Paying attention to that fact that every 1-dim **M**-**M** bimodule Y induces an automorphism θ of **M** such that $Y \cong {}^{\theta}L^2(\mathbf{M})$, as **M**-**M** bimodules, we turn to find the explicit form of the related automorphisms corresponding to the 1-dim bimodules in the graph.

First notice since $id, \hat{\alpha}, \pi_1, \pi_2$ appear in $X \otimes X^*$, they form the Galois group $Gal(\mathbf{M}, \mathbf{N})$. Indeed, we will eventually have $\Gamma(\mathbf{M}, \mathbf{N}) = Gal(\mathbf{M}, \mathbf{N})$ in our case. For this, let $\pi \in Gal(\mathbf{M}, \mathbf{N})$, and let λ the unitary implementing the automorphism α. T hen $\mathbf{M} = \mathbf{P} \times_\alpha K = \{a + b\lambda \mid a, b \in \mathbf{P}\}$. For the proper form of π, recall that $\alpha\beta = \beta\alpha \circ Adu$. Since

$$\lambda \cdot \beta(x) \cdot \lambda = \beta\alpha(uxu^*) = \beta(\alpha(u)\alpha(x)\alpha(u^*)) = u \cdot \beta\alpha(x) \cdot u^*,$$
$$Ad(u^*\lambda) : \beta(\mathbf{P}) \longrightarrow \beta\alpha(\mathbf{P}) \subseteq \beta(\mathbf{P})$$

is an automorphism of $\beta(\mathbf{P})$. Therefore for a trivial character of H,

$$\pi(a + b\lambda) = \beta(a) + \beta(b)u^*\lambda$$

determines an automorphism of $\mathbf{M}$ satisfying $\pi|_N(a) = \beta(a) = a$, due to $\mathbf{N} = \mathbf{P}^\beta$. i.e. $\pi \in Gal(\mathbf{M}, \mathbf{N})$ and $\pi|_P = \beta$. Moreover,

$$\pi^2(a + b\lambda) = \pi(\beta(a) + \beta(b)u^*\lambda) = \beta^2(a) + \beta^2(b)\beta(u^*)u^*\lambda = a + \omega b\lambda.$$

Thus, we can conclude that $\pi^2(a+b\lambda) = a+b\lambda$ if $\omega = 1$, $\pi^2(a+b\lambda) = a-b\lambda$ if $\omega = -1$. i.e. π has order 2 if $\omega = 1$, has order 4 if $\omega = -1$.

Proposition 2.1 *The 1-dim bimodules appearing in the dual principal graph (and hence the elements of the Galois group) are* $\{id, \hat{\alpha}, \pi, \hat{\alpha}\pi\}$.

Proof. We know $\pi \in Gal(\mathbf{M}, \mathbf{N})$. Note that the automorphism π of $\mathbf{M}$ yields a $\mathbf{M}$-$\mathbf{M}$ bimodule ${}^\pi L^2(\mathbf{M})$. As a $\mathbf{P}$-$\mathbf{P}$ bimodule,

$$B\pi B^* =_P L^2(\mathbf{M}) \otimes_M {}^\pi L^2(\mathbf{M}) \otimes_M L^2(\mathbf{M})_P =_P {}^\beta L^2(\mathbf{M})_P,$$

due to $\pi|_P = \beta$. But this contains $\mathbf{P}$-$\mathbf{P}$ bimodule $\beta =_P {}^\beta L^2(\mathbf{P})_P$ as an irreducible submodule, or β is an irreducible submodule of $B\pi B^*$. Therefore, by Frobenius reciprocity, we see that π is an irreducible submodule of $B^*\beta B$. Similarly, $\hat{\alpha}\pi(\neq \pi$, due to the outerness of $\hat{\alpha})$ is also an irreducible submodule of $B^*\beta B$. Since $B^*\beta B$ is a 2-dim bimodule, we must have $B^*\beta B = \pi \oplus \hat{\alpha}\pi$. This completes the proof. □

Indeed, it is easily see that $\hat{\alpha}\pi(a + b\lambda) = \beta(a) - \beta(b)u^*\lambda$, which occurs when the character of H is non trivial.

Consequently, $X^* \otimes X = id \oplus \hat{\alpha} \oplus \pi \oplus \hat{\alpha}\pi$, and the Galois group $\{id, \hat{\alpha}, \pi, \hat{\alpha}\pi\}$ is the same as $\Gamma(\mathbf{M}, \mathbf{N})$, in other words, every automorphism of $\mathbf{M}$ belongs to $Aut(\mathbf{M}, \mathbf{N})$.

Furthermore, we see that the Connes obstruction determine the Galois group in this case. For example, if Connes obstruction of $\alpha\beta$ is 1, then π, and hence $\hat{\alpha}\pi$ are of order 2. Thus the group must be Klein 4-group $Z_2 \oplus Z_2$. If Connes obstruction is -1, then π has of order 4, so the group must be the cyclic group Z_4.

It is worth to point out that the shape of princiapl graph does not remember Connes obstruction, while fusion rules actually see the Connes obstruction. In depth two cases, this will determine the group structure of cocycle-twisted Majid type Hopf algebras. From now on, we are going to write down the explicit fusion rules in our case, which even leads us to learn how Connes obstruction enters the group structure when the outer period is greater than 2.

Lemma 2.2 *Let* $Hom(H,\mathcal{T}) = H/[H,H] = \hat{H}$, *and Weyl group* $\mathcal{N}_G(H)/H$. *Then there exists an extension*

$$I \to \hat{H} \to Gal(\mathbf{M},\mathbf{N}) \to \mathcal{N}_G(H)/H \to I.$$

Proof. Let $\theta \in \mathcal{N}_G(H)/H$ where $H = \{1,\alpha\}$. Since $\theta\alpha\theta^{-1} = \alpha$ in $Out(\mathbf{P})$, there exists a unitary $v(\alpha,\theta) \in \mathbf{P}$ such that $Adv(\alpha,\theta)\theta\alpha\theta^{-1} = \alpha$. Since $\beta\alpha\beta^{-1} = Adu \circ \alpha$, in other words, β normalizes H in $Out(\mathbf{P})$, the Weyl group $\mathcal{N}_G(H)/H = \{e,\beta\}$ in our case, and hence we obtain the following.

$v(\cdot,\cdot)$	e	β
1	1	1
α	1	u^*

Since for $\theta \in \mathcal{N}_G(H)/H$,

$$Adv(hg,\theta)\theta\alpha_{hg}\theta^{-1} = \alpha_{hg} = Ad(v(h,\theta)\theta\alpha_h\theta^{-1}(v(g,\theta))\theta\alpha_{hg}\theta^{-1},$$

there exists 2-cocycle $\xi_\theta(h,g)$ of H such that

$$v(hg,\theta) = \xi_\theta(h,g)\alpha(v(g,\theta))v(h,\theta).$$

Clearly $\xi_e = 1$. Also $\xi_\beta(1,1) = \xi_\beta(1,\alpha) = 1$ and

$$\xi_\beta(\alpha,1) = 1, \quad \xi_\beta(\alpha,\alpha) = \alpha(u^*)u^*v(1,\beta) = 1.$$

i.e., the 2-cocycle ξ_θ of H is coboundary for all $\theta \in \mathcal{N}_G(H)/H$. Therefore we conclude that there exists an extension

$$I \to \hat{H} \to Gal(\mathbf{M},\mathbf{N}) \to \mathcal{N}_G(H)/H \to I,$$

since $\beta\alpha\beta^{-1} = \alpha$ in $Out(\mathbf{P})$ again implies that $\mathcal{N}_G(H)/H = \{e,\beta\} = Z_2$ acts trivially on $\hat{H} = Z_2$. □

To determine the structure of the group $Gal(\mathbf{M},\mathbf{N})$, we have to look at the fusion rules among automorphisms. For $\chi \in \hat{H}$ and $\theta \in \mathcal{N}_G(H)/H$, first we write the automorphism $\pi_{\chi,\theta}$ as

$$\pi_{\chi,\theta}(a+b\lambda) = \theta(a) + \chi(\alpha)\theta(b)v(\alpha,\theta)^*\lambda.$$

Indeed, denoting $\hat{H} = \{i, \chi\}$ with a non-trivial character χ satisfying $\chi(\alpha) = -1$, we have

$$\begin{aligned} \pi_{i,e}(a+b\lambda) &= a+b\lambda, \\ \pi_{\chi,e}(a+b\lambda) &= a-b\lambda, \\ \pi_{i,\beta}(a+b\lambda) &= \beta(a)+\beta(b)u^*\lambda, \\ \pi_{\chi,\beta}(a+b\lambda) &= \beta(a)-\beta(b)u^*\lambda, \end{aligned}$$

and each of which corresponds to an automorphism appearing in $X^* \otimes X$. From the construction, we see that

$$id = \pi_{i,e}, \quad \hat{\alpha} = \pi_{\chi,e}, \quad \pi = \pi_{i,\beta}, \quad \hat{\alpha}\pi = \pi_{\chi,\beta},$$

and hence $Gal(\mathbf{M}, \mathbf{N}) = \{\pi_{i,e}, \pi_{\chi,e}, \pi_{i,\beta}, \pi_{\chi,\beta}\}$. As we see, the whole Galois group is parametrized by the product Weyl group $\mathcal{N}_G(H)$ by the character group $Hom(H, \mathcal{T}) = H/[H,H] = \hat{H}$, and the fusion rule on $Gal(\mathbf{M}, \mathbf{N})$ depends on the 2-cohomology group $H^2(\mathcal{N}_G(H)/H, \hat{H}) = H^2(Z_2, Z_2) = Z_2$.

Proposition 2.2 *The Galois group $Gal(\mathbf{M}, \mathbf{N})$ is Klein 4-group $Z_2 \oplus Z_2$ when Connes obstruction is 1, and the cyclic group Z_4 when Connes obstruction is -1.*

Proof. First notice that for θ, $\theta' \in \mathcal{N}_G(H)/H$,

$$Ad(v(h, \theta\theta')^*)\alpha_h = \theta\theta'\alpha_h(\theta\theta')^{-1} = Ad[\theta(v(h,\theta')^*)v(h,\theta)^*]\alpha_h.$$

Thus there exists $\hat{\chi}_{\theta,\theta'} \in \mathcal{T}$ such that

$$\hat{\chi}_{\theta,\theta'}(h) = v(h, \theta\theta')\theta(v(h,\theta')^*)v(h,\theta)^*,$$

which turns out to be a 2-cohomology $\hat{\chi} \in H^2(\mathcal{N}_G(H)/H, \hat{H})$. Furthermore, an easy computation in our case gives

$$\hat{\chi}_{e,e} = \hat{\chi}_{e,\beta} = \hat{\chi}_{\beta,e} = 1, \text{ and } \hat{\chi}_{\beta,\beta}(1) = 1, \quad \hat{\chi}_{\beta,\beta}(\alpha) = \bar{\omega} = \omega.$$

i.e. χ is trivial if $\omega = 1$, χ is non-trivial if $\omega = -1$. Also we have

$$\pi_{\chi,\theta}\pi_{\chi',\theta'} = \chi_{\chi\chi'\hat{\chi}_{\theta,\theta'},\, \theta\theta'}.$$

With help of these fusion rules, the multiplication table for the the group $Gal(\mathbf{M}, \mathbf{N})$ is obtained as follows;

	$\pi_{i,e}$	$\pi_{\chi,e}$	$\pi_{i,\beta}$	$\pi_{\chi,\beta}$
$\pi_{i,e}$	$\pi_{i,e}$	$\pi_{\chi,e}$	$\pi_{i,\beta}$	$\pi_{\chi,\beta}$
$\pi_{\chi,e}$	$\pi_{\chi,e}$	$\pi_{i,e}$	$\pi_{\chi,\beta}$	$\pi_{i,\beta}$
$\pi_{i,\beta}$	$\pi_{i,\beta}$	$\pi_{\chi,\beta}$	$\pi_{\chi,e}$	$\pi_{i,e}$
$\pi_{\chi,\beta}$	$\pi_{\chi,\beta}$	$\pi_{i,\beta}$	$\pi_{i,e}$	$\pi_{\chi,e}$

$Gal(\mathbf{M},\mathbf{N}) = Z_4$, when Connes obstruction $\omega = 1$

	$\pi_{i,e}$	$\pi_{\chi,e}$	$\pi_{i,\beta}$	$\pi_{\chi,\beta}$
$\pi_{i,e}$	$\pi_{i,e}$	$\pi_{\chi,e}$	$\pi_{i,\beta}$	$\pi_{\chi,\beta}$
$\pi_{\chi,e}$	$\pi_{\chi,e}$	$\pi_{i,e}$	$\pi_{\chi,\beta}$	$\pi_{i,\beta}$
$\pi_{i,\beta}$	$\pi_{i,\beta}$	$\pi_{\chi,\beta}$	$\pi_{i,e}$	$\pi_{\chi,e}$
$\pi_{\chi,\beta}$	$\pi_{\chi,\beta}$	$\pi_{i,\beta}$	$\pi_{\chi,e}$	$\pi_{i,e}$

$Gal(\mathbf{M},\mathbf{N}) = Z_2 \oplus Z_2$, when Connes obstruction $\omega = -1$

This completes the proof. □

3 Nonadjustable automorphisms appearing in the dual principal graph

We begin with well known result which will help to find the counter-examples. Let $\mathbf{N} \subset \mathbf{M}$ be an irreducible subfactors with finite Jones index and finite depth. We denote by X the corresponding $\mathbf{N}$-$\mathbf{M}$ bimodule.

Proposition 3.1 ([5]) *An automorphism $\theta \in Aut(\mathbf{M})$ can be adjusted to an automorphism in $Aut(\mathbf{M},\mathbf{N})$ if and only if there exists an automorphism $\alpha \in Aut(\mathbf{M})$ satisfying $\theta \otimes L^2(\mathbf{M}) = L^2(\mathbf{M}) \otimes \alpha$ as bimodules. Furthermore, if θ is submodule appearing in the irreducible decomposition of $(X \otimes X^*)^n$ if and only if α is a submodule appearing in the decomposition of $(X^* \otimes X)^n$, for $n \geq 1$.*

Example 3.1 *Haagerup subfactor of index $\frac{5+\sqrt{13}}{2}$ explains this case ([1]). The principal graph has depth 5, its dual has depth 6 (indeed, 3-star graph T(4,4,4)). There exists only one non-trivial 1-dim bimodule in the principal* *while there exist two non-trivial 1-dim bimodules appearing the dual principla graph. Since any of those which appear in the dual donot appear in the second level $X \otimes X^*$, they can not be adjusted as one of $Aut(\mathbf{M}, \mathbf{N})$ thanks to Proposition 3.1.*

Another one can be obtained by considering a pair of subfactors constructed from the outer action of the alternating groups $A_4 \subset A_5$. Note that every automorphism in the crossed product algebras (and fixed point algebras) has an expression via the character of the acting group, for example

$$\theta_h(\sum_g x_g \lambda_g) = \sum_g \chi(hgh^{-1}) x_{hgh^{-1}} \lambda_{hgh^{-1}}$$

determines the automorphism of $\mathbf{P} \times G$.

Example 3.2 *Let the alternating groups $A_4 \subset A_5$ act outerly on $\mathbf{P}$. Let*

$$\mathbf{N} = \mathbf{P}^{A_5} \subset \mathbf{P}^{A_4} = \mathbf{M}.$$

If there exists an automorphism of $\mathbf{M}$ which can be adjusted as an element in $Aut(\mathbf{M}, \mathbf{N})$, the resulted automorphism of the form related to the character of A_4 should be extended to an automorphism of the formrelated to A_5. However, any non-trivial character of A_4 can be extended to a character of A_5, since the simple group A_5 has no non-trivial characters.

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