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A STUDY ON THE GENDERALIZED NONLINEAR COMPLEMENTERITY PROBLEM

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1. Introduction

The nonlinear complementarity problem (CP) is wellknown. It can be stated as follows.

(CP): Given a mapping $f:R_+^n \rightarrow R$, find an n-vector x_0 such that

$x_0 \in R_+^n$, $f(x_0) \in R_+^n$, and $\langle x_0, f(x_0)\rangle = 0$,

Several authors including Eaves([2]), Karamardian([3],[4]), and N. Megiddo and M. Kojima([5]) have studied existence and uniqueness theorems for (CP).
In this paper, We consider the following generalized nonlinear complementarity problem (GCP);

(GCP): Let C be a closed convex cone in R^n and C^* be the positive polar cone of C. Given a mapping $f:C \rightarrow R^n$, find an n-vector x_0 such that

$$x_0 \in C,\ f(x_0) \in C^*, \text{ and } \langle x_0, f(x_0)\rangle = 0,$$

with establishing existence and uniquenss theorems for (GCP).

2. Preliminaries

Let C be a closed convex cone in R, C^* be the positive polar cone of C and $f:C \rightarrow R^n$ be a mapping.

Definition 2.1. f is said to be *strictly* monotone if $\langle x-y, f(x) - f(y)\rangle \geq 0$ for all $x, y \in C$ and strict inequality holds whenever $x \neq y$.

Definition 2.2. f is called ***strongly*** monotone if there is a constant $c>0$ such that

$$\langle x-y, f(x) - f(y)\rangle \geq c\|x-y\|^2. \text{ for all } x,y \in C.$$

Definition 2.3 f is said to be ***Lipschitzian*** if there is a constant $k>0$ such that $\|f(x) - f(y)\| \leq k\|x-y\|$ for all $x,y \in C$.

Definition 2.4 f is said to be ***hemicontinuous*** if for all $x,y \in C$, the map $t \rightarrow f([ty + (1-t)x]$ of $[0,1]$ to R^n is contimuous.

Definition 2.5 f is said to be ***bounded*** if there is a constant $k>0$ such that $\|f(x)\| \leq k\|x\|$ for all $x \in C$.

Lemma 2.1 ([1]). Let $f: C \rightarrow R^n$ be hemicontinuous,strictly monotone and bounded and let $\{V_r\}$ be a family of nonempty closed convex sets in C. Then, for each r, there is a unique $x_r \in V_r$ such that $\langle x_r, f(x)\rangle \leq \langle z, f(x_r)\rangle$ for all $z \in V_r$.

3. Main Results

Now we established existence and uniqueness theorems for (GCP) under certain assumptions.

THEOREM 3.1. Let $f: C \rightarrow R^n$ be hemicontinuous,strictly monotone and bounded. Then 0 is the unique solution of (GCP).

Proof. For each $r \geq 0$, we write $B_r = \{x \in C: \|x\| \leq r)$. B_r is a nonempty closed set in C.

By Lemma 2.1, for each $r \geq 0$ there is a unique $x_r \in B_r$ such that $\langle x_r, f(x_r)\rangle \leq \langle z, f(x_r)\rangle$ for all $z \in B_r$. Since $0 \in B_r$ $\langle x_r, f(x_r)\rangle \leq 0$. We can define a function θ from $[0,\infty)$ to $(-\infty,0]$ by the rule $\theta(r) = \langle x_r, f(x_r)\rangle$. Now suppose that $r \neq 0$ and $r < s$. Then there are unique $x_r \in B_r$ and $x_s \in B_s$ such that

$$\langle x_r, f(x_r)\rangle \leq \langle z, f(x_r)\rangle \text{ for all } z \in B_r.$$

and

$$\langle x_s, f(x_s)\rangle \leq \langle z, f(x_s)\rangle \text{ for all } z \in B_s.$$

Since $(r/s)x_s \in B_r$, $\langle x_r, f(x_r)\rangle \leq (r/s)\langle x_s, f(x_r)\rangle$. Since $(s/r)x_r \in B_s$, $\langle x_s, f(x_s)\rangle \leq (s/r)\langle x_r, f(x_s)\rangle$. Hence we have

$$\langle x_r - x_s, f(x_r)\rangle = \langle x_r, f(x_r)\rangle + \langle x_s, f(x_s)\rangle - \langle x_s, f(x_r)\rangle - \langle x_r, f(x_s)\rangle$$

$$\le \langle x_r, f(x_r)\rangle + \langle x_s, f(x_s)\rangle - (s/r)\langle x_r, f(x_r)\rangle - (r/s)\langle x_s, f(x_s)\rangle$$
$$= [1-(s/r)]\theta(r) + [1-(r/s)]\theta(s)$$
$$= (s-r)\{[\theta(s)/s] - [\theta(r)/r]\}$$

Since $s > r$ and f is monotone, $\theta(s)/s \ge \theta(r)/r$. Therefore $\theta(r)/r$ is monotonically increasing on $(0,\infty)$. Since f is bounded, $|\theta(r)| = \langle x_r, f(x_r)\rangle \le \|x_r\| \cdot \|f(x_r)\| \le k\|x_r\|^2$. Hence $|\theta(r)| \le k r^2$. Since $\theta(r) < 0, -\theta(r) \le k r^2$, Consequently,

$-kr < \theta(r)/r \le 0$ for all $r \in (0,\infty)$.Since $\lim_{r\to 0+} [\theta(r)/r] = 0$ and $\theta(r)/r$ is monotonically increasing, it follows that $\theta(r)=0$ and hence $\theta(r)=0$ for all $r \in (0,\infty)$. So we have $\langle z, f(x_r)\rangle \ge 0$ for all $z \in B_r$.Since c is a cone,$\langle z, f(x_r)\rangle \ge 0$ for all $z \in c$. Therefore, for each $r \in (0,\infty)$,x_r is a solution of (GCP).

Now f is strictly monotone, (GCP) can have at most one solution, say x_0. $x_0 = x_r \in B_r$ for each r and $\|x_0\| = \|x_r\| \le r$ for each r. So $x_0 = 0$.

Corollary 3.1. Let $f: R_+^n \to R$ be hemicontinuous, strictly monotone and bounded. Then 0 is the unique solution of (CP).

Proof. R_+^n is a closed convex cone in R^n. By Theorem 3.1, the above result holds.

Theorem 3.2. Let $f: c \to R^n$ be strongly monotone and Lipschitzian with $k^2 < 2c < k^2+1$. Then there is the unique solution of (GCP).

Proof. Since c is a nonempty closed convex set in R^n, for every $x \in c$ there is a unique $y \in c$ closest to $x - f(x)$; that is $\|y - x + f(x)\| \le \|z - x + f(x)\|$ for all $z \in C$.

Let the correspondence $x \to y$ be denoted by θ. Let z be any element of C and let $0 \le \lambda \le 1$.

Since C is convex, $(1-\lambda)y + \lambda z \in C$. We define a map $h:[0.1] \to R_+$ by the rule

$$h(\lambda) = \|x - f(x) - (1-\lambda)y - \lambda z\|^2.$$

Then h is a twice continuously differentiable function of λ and $h'(\lambda) = 2\langle x - f(x) - \lambda z - (1-\lambda)y,\ y - z\rangle$. Since y is the unique element closet to $x - f(x)$, $h'(a) \ge 0$. So we have

(1) $\langle x - f(x) - y,\ y - z\rangle \ge 0$ for all $z \in C$.

Let x_1 and x_2 be two elements of C with $x_1 \neq x_2$. Put $\theta(x_1) = y_1$ and $\theta(x_2) = y_2$.

From (1), we get

$\langle x_1 - f(x_1) - \theta(x_1), \theta(x_1) - \theta(x_2)\rangle \geq 0$

and

$\langle x_2 - f(x_2) - \theta(x_2), \theta(x_2) - \theta(x_1)\rangle \geq 0$

From these two inequalities, we have

$\langle x_1 - f(x_1) - \theta(x_1) - x_2 + f(x_2) + \theta(x_2),$
$\theta(x_1) - \theta(x_2)\rangle \geq 0$

Hence,

$\langle x_1 - f(x_1) - x_2 + f(x_2), \theta(x_1) - \theta(x_2)\rangle \geq \langle \theta(x_1) - \theta(x_2), \theta(x_1) - \theta(x_2)\rangle =$
$\|\theta(x_1) - \theta(x_2)\|^2,$

Therefore,

$$\|\theta(x_1) - \theta(x_2)\|^2 \leq |\langle x_1 - f(x_1) - x_2 + f(x_2), \theta(x_1) - \theta(x_2)\rangle|$$
$$\leq \|x_1 - f(x_1) - x_2 + f(x_2)\| \cdot \|\theta(x_1) - \theta(x_2)\|$$

Thus, $\|\theta(x_1) - \theta(x_2)\| \leq \|f(x_1) - f(x_2) - x_1 + x_2\|$.

Since f is strongly monotone and Lipschitzian, we have

$$\|\theta(x_1) - \theta(x_2)\|^2 \leq f(x_1) - f(x_2) x_1 + x_2\|^2$$
$$= \langle f(x_1) - f(x_2) - x_1 + x_2, f(x_1) - f(x_2) - x_1 + x_2\rangle$$
$$= \|f(x_1) - f(x_2)\|^2 + \|x_1 - x_2\|^2$$
$$-2\langle x_1 - x_2, f(x_1) - f(x_2)\rangle$$
$$\leq k^2\|x_1 - x_2\|^2 + \|x_1 - x_2\|^2 - 2c\|x_1 - x_2\|^2$$
$$= (k^2 + 1 - 2c)\|x_1 - x_2\|^2$$

Since $k^2 < 2c < k^2+1$, we have $0 < k^2+1-2c < 1$.

Letting $\alpha = k^2+1-2c$ in the above inequality, we obtain

$\|\theta(x_1) - \theta(x_2)\| \leq \alpha\|x_1 - x_2\|$ with $0 < \alpha < 1$.

By the Banach contraction principle, θ has the unique fixed point, say x_0.

Now putting $x = x_0$ in (1), We get $\langle z - x_0, f(x_0)\rangle > 0$ for all $z \in C$. Since $0 \in C$, $\langle x_0, f(x_0)\rangle \leq 0$. Since C is a cone, $2x_0 \in C$ and $\langle x_0, f(x_0)\rangle \geq 0$.

So $\langle x_0, f(x_0)\rangle = 0$.

and $\langle z, f(x_0)\rangle > 0$ for all $z \in C$. Therefore, x_0 is the unique solution of (GCP).

Corollary 3.2. Let $f: R^n_+ \to R$ be strongly monotone and Lipschitzian with

$k^2 < 2C < k^2+1$.

Then there is the unique solution of (CP).

Proof. R^n_+ is a closed convex cone in R. By Therem 3.1, the above result holds.

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A. 연구소 동정

1. 최일동 교수가 1992년 9월 1일자로 해양재료공학과에 부임하시다.

B. 연구원 활동

1. 이철영교수가 1992년 10월 17일 퍼지시스템학회 부산·경남지회장에 피선되시었고, 박춘일 교수는 감사에 피선되시다.

2. 1992년 10월 13일 부산대학교 통계학과 주최로 영국의 John Nelder교수의 일반선형모형에 대한 Work Shop에 박춘일 교수가 좌장을 맡아보시다.

3. 금상호 교수가 1992년 7월 10일에 개최된 서울대학교 대역해석학 연구센터의 세미나에 참석하여 연구논문을 발표하시다.

C. 연구원 동정

1. 김장욱 교수가 1992년 3월 1일자로 기초과학연구소장으로 부임하시다.

2. 박춘일 교수가 1992년 3월 1일자로 응용수학과장으로 부임하시다.