DUAL OPERATOR ALGEBRAS AND HEREDITARY PROPERTIES OF MINIMAL JOINT DILATIONS

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1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A dual algebra is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. The theory of dual algebras is deeply related to the study of the problem of solving systems of simultaneous equations in the predual of a dual algebra (see [1], [3] and [4]). In particular, Exner-Jung [11] defined certain Hereditary properties concerning a minimal isometric dilation of a contraction operator $T$ and obtained some characterizations for membership in the the class $\mathcal{A}_{1,\infty_0}$ which will be defined below. This give a motivation for this work.

Let $T$ be a contraction operator in $\mathcal{L}(\mathcal{H})$ and let $B_T$ be a minimal isometric dilation of $T$ on $\mathcal{K}_+$,

$$\mathcal{K}_+ = \bigvee_{n=0}^{\infty} B_T^n \mathcal{H},$$

with the Wold decomposition $B_T = S_T \oplus R_T$, where $S_T \in \mathcal{L}(\mathcal{U}_T)$ is the unilateral shift part and $R_T \in \mathcal{L}(\mathcal{R}_T)$ is the residual part. Suppose that $T \in \mathcal{L}(\mathcal{H})$ has a non-zero semi-invariant subspace $\mathcal{M}$ (i.e., $\mathcal{M} \neq \{0\}$). For a compression $\tilde{T} = T_{\mathcal{M}}$ of $T$ to $\mathcal{M}$, we write a minimal isometric dilation of $\tilde{T}$ by $B_{\tilde{T}} = S_{\tilde{T}} \oplus R_{\tilde{T}}$. Recall that a contraction $T$ has property (H) if, for any non-zero semi-invariant subspace $\mathcal{M}$ for $T$, the minimal isometric dilation $B_{T_{\mathcal{M}}} \in \mathcal{L}(\tilde{\mathcal{K}})$ of $T_{\mathcal{M}}$ which is obtained as a restriction $B_T|\tilde{\mathcal{K}}$ with $\tilde{\mathcal{K}} \in \text{Lat}(B_T)$ satisfies $\mathcal{U}_{T_{\mathcal{M}}} \subset \mathcal{U}_T$. In addition, a contraction operator $T \in \mathcal{A}$ has property (P) if there exists $\mathcal{M} \in \text{Lat}(T)$ such that $T|\mathcal{M} \in \mathcal{A}(\mathcal{M})$ and

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T|\mathcal{M} has property (P). Then it follows from \cite{[1]} that $T \in A_{1,n}$ if and only if $T$ has property (P). Also one discuss other related hereditary properties in \cite{[15]}. In \cite{[9]}, one developed a functional calculus for a 2-tuple contractions. In \cite{[4]}, one introduced a class $A_{m,n}$ of pairs of operators and obtained some results concerning minimal joint isometric dilations. So in this paper we extend the study of hereditary properties of single operators to 2-tuple operators.

In section 3 we will discuss certain Hereditary properties concerning minimal isometric dilations or minimal coisometric extensions of a pair of contractions. In section 3, we apply the hereditary properties to a theory of dual algebras.

2. Preliminaries

The notation and terminology employed here agree with those in \cite{[2]}, \cite{[4]} and \cite{[19]}. Suppose that $\mathcal{A}$ is a dual algebra in $\mathcal{L}(\mathcal{H})$. Let $C_1 = C_1(\mathcal{H})$ be the trace class in $\mathcal{L}(\mathcal{H})$ and let $\uparrow A$ denote the preannihilator of $A$ in $C_1$. Let $\mathcal{Q}_A$ denote the quotient space $C_1/\uparrow A$. One knows that $A$ is the dual space of $\mathcal{Q}_A$ and that the duality is given by

$$< T, [L] > = \text{trace}(TL), \ T \in \mathcal{A}, \ [L] \in \mathcal{Q}_A.$$

For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the dual algebra generated by $T$. For vectors $x$ and $y$ in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in $C_1$ defined by $(x \otimes y)(u) = (u, y)x, \ u \in \mathcal{H}$.

Throughout this paper, we write $\mathbb{N}$ for the set of natural numbers. We shall denote by $\mathbb{D}$ the open unit disc in the complex plane $\mathbb{C}$ and we write $\mathbb{T}$ for the boundary of $\mathbb{D}$.

For a Hilbert space $\mathcal{K}$ and any operators $T_i \in \mathcal{L}(\mathcal{K}), \ i = 1, 2$, we write $T_1 \cong T_2$ if $T_1$ is unitarily equivalent to $T_2$.

For $1 \leq p \leq \infty$ we denote the usual Lebesgue function space by $L^p = L^p(\mathbb{T})$ and the usual Hardy space by $H^p = H^p(\mathbb{T})$. One knows that the preannihilator $\uparrow (H^\infty)$ of $H^\infty$ in $L^1$ is the subspace $H^1_0$ consisting of those functions $g$ in $H^1$ for which analytic extension $\widehat{g}$ to $\mathbb{D}$ satisfies $\widehat{g}(0) = 0$ (cf. \cite{[14]}). It is well known that $H^\infty$ is the dual space of $L^1/H^1_0$.

Suppose that $m$ and $n$ are any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(A_{m,n})$ if every $m \times n$ system of simultaneous equations of the form $[x_i \otimes y_j] = [L_{ij}], \ 0 \leq i < m, \ 0 \leq j < n$, where $\{(L_{ij})\}_{0 \leq i < m, 0 \leq j < n}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_A$, has a solution $\{x_i\}_{0 \leq i < m}, \ \{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. For brevity, we shall denote $(A_{m,n})$ by $(A_n)$. The class $\mathcal{A}(\mathcal{H})$ consists of all
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those absolutely continuous contractions $T$ in $\mathcal{L}(\mathcal{H})$ for which the Foias-Nagy functional calculus $\Phi_T : H^\infty \rightarrow \mathcal{A}_T$ is an isometry. Furthermore, we denote by $\mathcal{A}_{m,n}(\mathcal{H})$ the set of all $T$ in $\mathcal{A}(\mathcal{H})$ such that the algebra $\mathcal{A}_T$ has property $(\mathcal{A}_{m,n})$. We write simply $\mathcal{A}_{m,n}$ for $\mathcal{A}_{m,n}(\mathcal{H})$ unless we mention otherwise.

Let $\mathcal{L}(\mathcal{H})_{\text{comm}}^{(2)}$ be the algebra of pairs of operators in $\mathcal{L}(\mathcal{H})$ which are commute. For $\mathbf{T} = (T_1, T_2) \in \mathcal{L}(\mathcal{H})_{\text{comm}}^{(2)}$, if there exists a 2-tuple $(S_1, S_2) \in \mathcal{L}(\mathcal{K})_{\text{comm}}^{(2)}$ for some Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\mathcal{H}$ is a semi-invariant subspace for $S_j$ and $(S_j)_{\mathcal{H}} = T_j, j = 1, 2$. For $\mathbf{T} = (T_1, T_2) \in \mathcal{L}(\mathcal{H})_{\text{comm}}^{(2)}$, a joint dilation(extension) $\mathbf{S} = (S_1, S_2) \in \mathcal{L}(\mathcal{K})_{\text{comm}}^{(2)}$, where $\mathcal{K} \supset \mathcal{H}$, is said to be a joint unitary (resp. isometric, coisometric ) dilation (extension) if each of $S_j, j = 1, 2$, is a unitary (resp. an isometry, a coisometry). If $\mathbf{U} = (U_1, U_2) \in \mathcal{L}(\mathcal{K})_{\text{comm}}^{(2)}$ is a joint dilation for $\mathbf{T} = (T_1, T_2) \in \mathcal{L}(\mathcal{H})_{\text{comm}}^{(2)}$ and $\mathcal{K} \supset \mathcal{H}$ is a common invariant subspace for $\mathbf{U}$, then $\mathbf{U}|\mathcal{K}' = (U_1|\mathcal{K}', U_2|\mathcal{K}') \in \mathcal{L}(\mathcal{K})_{\text{comm}}^{(2)}$ is a joint dilation of $\mathbf{T}$.

For $1 \leq p \leq \infty$, we denote by $L^p(\mathbb{T}^2)$ the Lebesgue spaces relative to normalized Lebesgue area measure on the torus $\mathbb{T}^2$ and by $H^p(\mathbb{T}^2)$ the Hardy space the subspaces of $L^p(\mathbb{T}^2)$ consisting of all the functions $f \in L^p(\mathbb{T}^2)$ such that the Foisson Kernel is analytic on $D^2$. We write for $L_0^1(\mathbb{T}^2)$ the subspace of $L^1(\mathbb{T}^2)$ consisting of those functions $f \in L^1(\mathbb{T}^2)$ such that Fourier coefficient $f(\mathbf{n}) = 0$, for all $n_1, n_2 \in \mathbb{N}$.

The following provides a good relationship between $H^\infty(\mathbb{T}^2)$ and $\mathcal{A}_{T_1,T_2}$ which is a dual algebra generated by $T_1, T_2$.

**Theorem 2.1.** [q. Theorem 2.4.2] If $(T_1, T_2) \in \text{ACC}^{(2)}(\mathcal{H})$, then there is an algebra homomorphism $\Phi_{T_1,T_2} : H^\infty(\mathbb{T}^2) \rightarrow \mathcal{A}_{T_1,T_2}$ with the following properties:

(a) $\Phi_{T_1,T_2}(1) = I_{\mathcal{H}}, \Phi_{T_1,T_2}(\omega_1) = T_1, \Phi_{T_1,T_2}(\omega_2) = T_2$, where $\omega_1$ and $\omega_2$ denote the coordinate functions.

(b) $\|\Phi_{T_1,T_2}(h)\| \leq \|h\|_\infty$, for all $h \in H^\infty(\mathbb{T}^2)$.

(c) $\Phi_{T_1,T_2}$ is weak*-continuous.(i.e., continuous when both $H^\infty$ and $\mathcal{A}_{T_1,T_2}$ are given the corresponding weak*-topologies).

(d) The range of $\Phi_{T_1,T_2}$ is weak*-dense in $\mathcal{A}_{T_1,T_2}$.

(e) There is a bounded, linear, one-to-one map

$$\phi_{T_1,T_2} : Q_{T_1,T_2} \rightarrow L^1(\mathbb{T}^2)/L^1_0(\mathbb{T}^2)$$

with $\phi_{T_1,T_2} = \Phi_{T_1,T_2}$.
(f) If $\Phi_{T_1,T_2}$ is an isometry, then it is a weak*-homeomorphism onto $\mathcal{A}_{T_1,T_2}$ and $\phi_{T_1,T_2}$ is an isometry onto $L^1(T^2)/L^1_0(T^2)$.

We now define the classes $\mathcal{A}^{(2)}(\mathcal{H})$ and $\mathcal{A}_{m,n}^{(2)}(\mathcal{H})$. The analogous classes $\mathcal{A}(\mathcal{H})$ and $\mathcal{A}_{m,n}(\mathcal{H})$ have been a central topic of study in the theory of dual algebras (cf. [4]). Let $(T_1, T_2) \in \text{AC}^{(2)}(\mathcal{H})$. We say that $(T_1, T_2) \in \mathcal{A}^{(2)}(\mathcal{H})$ if the functional calculus is an isometry. Furthermore, for $n \in \mathbb{N}$, we say that $(T_1, T_2) \in \mathcal{A}^{(2)}_{1/n}(\mathcal{H})$ if $(T_1, T_2) \in \mathcal{A}^{(2)}(\mathcal{H})$ and $\mathcal{A}_{T_1,T_2}$ has property $(\mathcal{A}_{1/n})$.

Similarly, for $m, n$ cardinal numbers with $1 \leq m, n \mathbb{N}$, we say that $\mathcal{A}_{m,n}^{(2)}(\mathcal{H})$ if $(T_1, T_2) \in \mathcal{A}^{(2)}(\mathcal{H})$ and $\mathcal{A}_{T_1,T_2}$ has property $(\mathcal{A}_{m,n})$. As before, we write $\mathcal{A}_n^{(2)}(\mathcal{H})$ instead of $\mathcal{A}_{1,n}^{(2)}(\mathcal{H})$.

Recall that $T \in C_0$ if $\|T^*x\| \to 0$ for any $x \in \mathcal{H}$. We say $T \in C_0$. if $T^* \in C_0$. And we denote that $C_{00} = C_0 \cap C_0$.

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Let $T = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ be a pair of contractions. A pair $(T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ of contractions has a joint coisometric extension, and thus a minimal joint coisometric extension (cf. [1]). Let $\mathcal{B}_T = (B_1, B_2) \in \mathcal{L}(\mathcal{K})^{(2)}_{\text{comm}}$ be a minimal joint isometric dilation of $T = (T_1, T_2)$, so that $\mathcal{K} \supset \mathcal{H}$, $\mathcal{H}$ is a common invariant subspace for $B_j$, and $B_j|\mathcal{H} = T_j^*$, $j = 1, 2$. Then $\mathcal{K}$ has decomposition $\mathcal{K} = S_j \oplus R_j$, $j = 1, 2$, such that $S_j, R_j$ are reducing subspaces for $B_j$, $j = 1, 2$ respectively and $B_j|S_j = S_j^*$, and $B_j|R_j = R_j$, $j = 1, 2$, where $S_j^*$ is backward shifts operator of some multiplicity and $R_j$ is unitary operator $j = 1, 2$. Furthermore, it follows that $\mathcal{B}_T^* = (B_1^*, B_2^*) \in \mathcal{L}(\mathcal{K})^{(2)}_{\text{comm}}$ is a minimal joint coisometric extension of $T^* = (T_1^*, T_2^*)$.

Let $\mathcal{M}$ be a common invariant subspace for $T = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ with $\mathcal{M} \neq (0)$. Then a minimal joint isometric dilation $\mathcal{B}_T = (B_1, B_2)$ of $T$ is a joint isometric dilation of $T|\mathcal{M} = (T_1|\mathcal{M}, T_2|\mathcal{M})$. Hence $T|\mathcal{M}$ has a minimal isometric dilation $\mathcal{B}_{T|\mathcal{M}} \in \mathcal{L}(\tilde{\mathcal{K}})_{\text{comm}}$ with $\mathcal{B}_{T|\mathcal{M}} = (B_1|\tilde{\mathcal{K}}, B_2|\tilde{\mathcal{K}})$ such that $\mathcal{M} \subset \tilde{\mathcal{K}} \subset \mathcal{K}$ with $\tilde{\mathcal{K}} \in \text{Lat}(\mathcal{B}_T)$ and $\mathcal{B}_{T|\mathcal{M}} = \mathcal{B}_T|\tilde{\mathcal{K}}$. Note that contrary to the one variable case, all minimal unitary dilation of a pair of contractions are not isometric. Hence we should define hereditary properties with a slight difference with one variable case as follows.

**Definition 3.1.** Suppose that $T = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}}$ be a pair of contractions.

(1) $T$ has property $(\mathcal{H}_1^{(2)})$ if there exists minimal joint isometric dilation $\mathcal{B}_T$ of $T$ such that for any non-zero common invariant subspace $\mathcal{M}$ for
$T = (T_1, T_2)$ the minimal joint isometric dilation $B_{T|\mathcal{M}} = (B_1|\tilde{\mathcal{K}}, B_2|\tilde{\mathcal{K}}) \in \mathcal{L}(\tilde{\mathcal{K}})_{\text{comm}}^{(2)}$ of $T|\mathcal{M}$ obtained as a restriction $B_{T|\tilde{\mathcal{K}}}$ with $\tilde{\mathcal{K}} \in \text{Lat}(B_T)$ satisfies $S_{T_1|\mathcal{M}} \subset S_{T_1}$ and $S_{T_2|\mathcal{M}} \subset S_{T_2}$.

(2) $T$ has property $(\text{H}^{(2)*}_1)$ if there exists minimal joint coisometric extension $B_T$ of $T$ such that for any non-zero common invariant subspace $\mathcal{M}$ for $T = (T_1, T_2)$, the minimal joint coisometric extension $B'_{T|\mathcal{M}} = (B'_1|\tilde{\mathcal{K}}, B'_2|\tilde{\mathcal{K}}) \in \mathcal{L}(\tilde{\mathcal{K}})_{\text{comm}}^{(2)}$ of $T|\mathcal{M}$ obtained as a restriction $B'_{T|\tilde{\mathcal{K}}}$ with $\tilde{\mathcal{K}} \in \text{Lat}(B'_T)$ satisfies $S_{T_1|\mathcal{M}} \subset S_{T_1}$ and $S_{T_2|\mathcal{M}} \subset S_{T_2}$.

(3) $T$ has property $(\text{H}_2^{(2)})$ if there exists minimal joint isometric dilation $B_T$ of $T$ such that for any non-zero common invariant subspace $\mathcal{M}$ for $T = (T_1, T_2)$ the minimal joint isometric dilation $B_{T|\mathcal{M}} = (B_1|\mathcal{M}, B_2|\mathcal{M}) \in \mathcal{L}(\tilde{\mathcal{K}})_{\text{comm}}^{(2)}$ of $T|\mathcal{M}$ obtained as a restriction $B_{T|\tilde{\mathcal{K}}}$ with $\tilde{\mathcal{K}} \in \text{Lat}(B_T)$ satisfies $\mathcal{R}_{T_1|\mathcal{M}} \subset \mathcal{R}_{T_1}$ and $\mathcal{R}_{T_2|\mathcal{M}} \subset \mathcal{R}_{T_2}$.

(4) $T$ has property $(\text{H}^{(2)*}_2)$ if there exists minimal joint coisometric extension $B_T$ of $T$ such that for any non-zero common invariant subspace $\mathcal{M}$ for $T = (T_1, T_2)$, the minimal joint coisometric extension $B'_{T|\mathcal{M}} = (B'_1|\tilde{\mathcal{K}}, B'_2|\tilde{\mathcal{K}}) \in \mathcal{L}(\tilde{\mathcal{K}})_{\text{comm}}^{(2)}$ of $T|\mathcal{M}$ obtained as a restriction $B'_{T|\tilde{\mathcal{K}}}$ with $\tilde{\mathcal{K}} \in \text{Lat}(B'_T)$ satisfies $\mathcal{R}_{T_1|\mathcal{M}} \subset \mathcal{R}_{T_1}$ and $\mathcal{R}_{T_2|\mathcal{M}} \subset \mathcal{R}_{T_2}$.

We denote $C_0^{(2)}$ is the set of pairs $(T_1, T_2)$ of operators on $\mathcal{H}$ with $T_i \in C_0(\mathcal{H})$ $i = 1, 2$.

**Proposition 3.2.** Suppose that $T = (T_1, T_2) \in C_0^{(2)}(\mathcal{H})$. Then $T$ has property $(\text{H}_1^{(2)})$.

**Proof.** Since $T_i \in C_0$, $B_{T_i}$ is a unilateral shift on $\mathcal{H}_i$. Let us consider a joint isometric dilation $B := (B_{T_1} \oplus B_{T_2}, B_{T_1} \oplus B_{T_2}) \in \mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)_{\text{comm}}^{(2)}$ of $T$.

Then $B \in C_0^{(2)}$ and there exists a minimal joint isometric dilation $B_T$ of $T$ such that $B|\mathcal{K} = B_T := (B'_1, B'_2)$. Since $B \in C_0^{(2)}$, $B_T \in C_0^{(2)}$. Consider $\mathcal{M} \in \text{Lat}(T)$ with $T|\mathcal{M} := (T_1|\mathcal{M}, T_2|\mathcal{M}) \in C_0^{(2)}$. Then since there is no unitary part, $T$ has property $(\text{H}_1^{(2)})$. $\square$

**Example 3.3.** Let $U$ be a bilateral shift and let consider $U = (U, U)$. Then $U = B_U$ and $S = (0)$. Let $\mathcal{M} \in \text{Lat}(U)$. Then $U|\mathcal{M} = B_U|\mathcal{M}$ and $\tilde{S} = \mathcal{M}$. Hence $\tilde{S}$ can not be contained in $S$ and $U$ does not have property $(\text{H}_1^{(2)})$.
**Theorem 3.4.** Every pair of contractions \( T = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}} \) has

1. property \( (H_1^{(2)^*}) \),
2. property \( (H_2^{(2)}) \),
3. property \( (H_2^{(2)^*}) \).

**Proof.** Let \( T = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{\text{comm}} \) and \( \mathcal{M} \) be a nontrivial common invariant subspace of \( T \).

1. Let \( B_T = (B_{T_1}', B_{T_2}') \), be the Ando coisometric extension of \( T \) and let \( B_T' = (B_{T_1}'', B_{T_2}'') \) be a minimal joint coisometric extension of \( T = (\tilde{T}_1, \tilde{T}_2) \) such that \( \tilde{K} \subset K \). Then

\[
B_{T_i}' = S_{T_i}^* \oplus R_{T_i}^* \in \mathcal{L}(S_{T_i} \oplus R_{T_i}), \quad i = 1, 2
\]

and

\[
B_{T_i}'' = S_{T_i}^{**} \oplus R_{T_i}^{**} \in \mathcal{L}(S_{T_i} \oplus R_{T_i}), \quad i = 1, 2
\]

relative to a decomposition \( \mathcal{M} \oplus (\tilde{K} \oplus \mathcal{M}) \).

We now claim that \( S_{\tilde{T}_i} \subset S_{T_i} \), \( i = 1, 2 \). Let \( x \in S_{\tilde{T}_i} \) and let \( x = s \oplus r \in S_{T_i} \oplus R_{T_i} \). Since \( B_{T_i}'' = B_{T_i}' \tilde{K} \), by minimality where \( \tilde{K} \in \text{Lat}(B_{T_i}') \), we have

\[
||S_{T_i}^{*n}x||^2 = ||B_{T_i}'^nx||^2 = ||B_{T_i}''x||^2
\]

\[
= ||(S_{T_i}^{*n} \oplus R_{T_i}^{*n})(s \oplus r)||^2
\]

\[
= ||S_{T_i}^{*n}s||^2 + ||R_{T_i}^{*n}r||^2
\]

\[
= ||S_{T_i}^{*n}s||^2 + ||r||^2
\]

let \( n \to \infty, r = 0 \). Therefore \( x \in S_{T_i} \), \( i = 1, 2 \). Hence \( T \) has property \( (H_1^{(2)^*}) \).

(3) Using notation in the proof of \( (H_1^*) \). We claim that \( R_{\tilde{T}_i} \subset R_{T_i} \). Let \( x \in R_{\tilde{T}_i} \subset K = S_{\tilde{T}_i} \oplus R_{T_i} \) and let \( x = s \oplus r \in S_{T_i} \oplus R_{T_i} \). Then we have

\[
||s||^2 + ||r||^2 = ||x||^2 = ||R_{\tilde{T}_i}^{*n}x||^2 = ||B_{T_i}'^nx||^2
\]

\[
= ||B_{T_i}'^nx||^2 = ||S_{T_i}^{*n}x||^2 + ||R_{T_i}^{*n}r||^2
\]

\[
= ||S_{T_i}^{*n}s||^2 + ||r||^2.
\]
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Let \( n \to \infty \), since \( \|S_{T_i}^n s\| \to 0 \), \( s = 0 \). Therefore \( \mathcal{R}_{\overline{T}_1} \subset \mathcal{R}_{T_1} \). Similarly \( \mathcal{R}_{\overline{T}_2} \subset \mathcal{R}_{T_2} \). Hence \( T \) has property \((H_2^{(2)})^*\).

(2) Let \( B_T \in \mathcal{L}(\mathcal{K})^{(2)}_{\text{comm}} \) be the Ando minimal joint isometric dilation of \( T \) and \( B_{\overline{T}} = B_T|_M \in \mathcal{L}(\overline{\mathcal{K}})^{(2)}_{\text{comm}} \) be the minimal joint isometric dilation of \( \overline{T} = T|_M = (T_1|_M, T_2|_M) \) such that \( B_T|_{\overline{\mathcal{K}}} = B_{\overline{T}} \) and \( \overline{\mathcal{K}} \in \text{Lat}(B_1, B_2) \).

We have \( B_{T_i} = S_{T_i} \oplus R_{T_i} \in \mathcal{L}(S_{T_i} \oplus R_{T_i}) \) and \( B_{\overline{T}_i} = S_{\overline{T}_i} \oplus R_{\overline{T}_i} \in \mathcal{L}(S_{\overline{T}_i} \oplus R_{\overline{T}_i}) \).

We shall claim \( \mathcal{R}_{\overline{T}_i} \subset \mathcal{R}_{T_i} \), \( i = 1, 2 \). Let \( x \in \mathcal{R}_{\overline{T}_i} \) and let \( x = s \oplus r \in S_{T_i} \oplus R_{T_i} \).

Since \( B_{\overline{T}_i} = B_{T_i}|_{\overline{\mathcal{K}}} \),

\[
B_{T_i}^{*n} = \begin{pmatrix} B_{T_i}^{*n} & 0 \\ A_n & * \end{pmatrix}
\]

relative to a decomposition \( \overline{\mathcal{K}} \oplus (\mathcal{K} \oplus \overline{\mathcal{K}}) \) for any \( n \in \mathbb{N} \).

\[
\|x\|^2 \leq \|x\|^2 + \|A_n x\|^2 + \|B_{T_i}^{*n} x\|^2 + \|A_n x\|^2 = \|B_{T_i}^{*n} x\|^2 = \|B_{T_i}^{*n} x\|^2 \\
\leq \|x\|^2.
\]

Therefore \( \|x\| = \|B_{T_i}^{*n} x\| \) for any \( n \in \mathbb{N} \). And we have

\[
\|s\|^2 + \|r\|^2 = \|x\|^2 = \|B_{T_i}^{*n} x\|^2 = \|S_{T_i}^{*n} s\|^2 + \|R_{T_i}^{*n} r\|^2 = \|S_{T_i}^{*n} s\|^2 + \|r\|^2
\]

Letting \( n \to \infty \), \( \|S_{T_i}^{*n} s\| \to 0 \), \( s = 0 \). Therefore \( x \in \mathcal{R}_{T_1} \). So \( \mathcal{R}_{\overline{T}_1} \subset \mathcal{R}_{T_1} \). Similarly \( \mathcal{R}_{\overline{T}_2} \subset \mathcal{R}_{T_2} \). Hence \( T \) has property \((H_2^{(2)})^*\). \( \square \)

4. DUAL ALGEBRAS GENERATED BY 2-TUPLE CONTRACTIONS

Lemma 4.1. If \( T = (T_1, T_2) \in A^{(2)}_{1,1}(\mathcal{H}) \), then for any positive integer \( n \), there exists \( \mathcal{M}_n \in \text{Lat}(T) \) and \( \{e_k^{(n)}\}_{k=1}^n \subset \mathcal{M}_n \) such that

\[
e_k^{(n)} \in \text{Ker}(T_i|\mathcal{M}_n)^*k \ominus \text{Ker}(T_i|\mathcal{M}_n)^*k-1
\]

and

\[
[e_k^{(n)} \otimes e_k^{(n)}] = [C_0]_T
\]
$k = 1, \cdots n.$

**Proof.** Let us consider the operator $A_j \in \mathcal{L}(\mathbb{C}^n)$, $j = 1, 2$ such that

$$A_j = \begin{pmatrix} 0 & 1 \\ 0 & \ddots \\ \cdots & & 1 \\ 0 & & & 0 \end{pmatrix}.$$

Then it is easy to show that $A_j$ has a cyclic vector. By [17. Theorem 3.3.1], there exist $\mathcal{M}_n, \mathcal{N}_n \in \text{Lat}(T)$ with $\mathcal{M}_n \supset \mathcal{N}_n$ and $\dim(\mathcal{M}_n \ominus \mathcal{N}_n) = n$ such that $A_i^\prime = T_i \mathcal{M}_n \ominus \mathcal{N}_n$ is similar to $A_i$. Let $X : \mathcal{M}_n \ominus \mathcal{N}_n \longrightarrow \mathcal{H}$ invertible operator with $A_i = X A_i^\prime X^{-1}$ for $i = 1, 2$, then $X A_i^\prime = A_i X$ and then $X^* A_i^* = A_i^* X^*$. Define $e_k^{(n)} = X^* u_k$, where $u_k$. Then

$$(A_i^*)^j e_k^{(n)} = A_i^* X^* u_k = X^* A_i^* u_k.$$  

If $j \leq k$, then $X^* A_i^* u_k = 0$. Since $X^*$ is one to one, if $j \leq k$, then $X^* A_i^* u_k \neq 0$, for $i = 1, 2$. Hence we have

$$e_k^{(n)} \in \text{Ker}(A_i^* k \ominus \text{Ker}(A_i^* k)^{-1}.$$

Moreover, since

$$(T_i | \mathcal{M}_n)^* = \begin{pmatrix} * & 0 \\ * & A_i^* k \end{pmatrix},$$

we have

$$e_k^{(n)} \in \text{Ker}(T_i | \mathcal{M}_n)^* \ominus \text{Ker}(T_i | \mathcal{M}_n)^{k-1}.$$  

Thus, for $h \in H^\infty(T)$,

$$h(T_1, T_2) = \begin{pmatrix} * & * & * \\ 0 & \text{D} & * \\ 0 & 0 & * \end{pmatrix},$$

where $D$ is the scalar operator (with respect to basis $\{u_n\}$) defined by $De_k^{(n)} = h(\lambda) e_k^{(n)}$, therefore

$$\langle h(T_1, T_2) e_k^{(n)}, e_k^{(n)} \rangle = h(\lambda).$$
Then by definition of

\[ [C_{\lambda}]_T = [e_k^{(n)} \otimes e_k^{(n)}]_T, \quad k = 1, \cdots n. \]

Hence the proof is complete. \( \square \)

Let \( \mathcal{A} \) be a dual algebra, for \( 0 \leq \theta < \gamma \leq 1 \). We recall from \( [\ ] \) that \( \mathcal{E}_{\theta}^r(\mathcal{A}) \) is the set of all \( [L] \) in \( Q_{\mathcal{A}} \) such that there exist sequences \( \{x_i\} \) and \( \{y_i\} \) in the unit ball of \( \mathcal{H} \) satisfying

\[ \limsup_{i \to \infty} \|[L] - [x_i \otimes y_i]\| \leq \theta \]

and

\[ \|[x_i \otimes z]\| \to 0, \quad \forall z \in \mathcal{H}. \]

Furthermore, for some \( 0 \leq \theta < \gamma \leq 1 \), \( \mathcal{A} \) has property \( E_{\theta, \gamma}^r \) if \( \overline{ac}(\mathcal{E}_{\theta}^r(\mathcal{A})) \cap B_0, \gamma = \{[L] \in Q : \|[L]\| \leq \gamma\} \).

**Theorem 4.2.** Suppose \( T = (T_1, T_2) \in A_{1,1}^{(2)}(\mathcal{H}) \) has property \( (H_1^{(2)}) \). Then there exist \( \{f_j\}_{j=1}^\infty \) of unit vectors in \( \mathcal{H} \) satisfying

\[ [f_j \otimes f_j] = [C_0]_T \]

and

\[ \|[f_j \otimes w_1]_{T_1} + [f_j \otimes w_2]_{T_2}\| \to 0 \quad \forall w_i \in \mathcal{H}, \quad i = 1, 2. \]

**Proof.** Let \( B_T = (B_{T_1}, B_{T_2}) \) be a minimal joint isometric dilation of \( T = (T_1, T_2) \) has property \( (H_1^{(2)}) \), and let \( B_T^* = (B_T^*, B_T^*) \) be a minimal joint co-isometric extension of \( T^* = (T_1^*, T_2^*) \). Suppose \( B_{T_1}^* = S_{T_1}^* \oplus R_{T_1}^* \), \( B_{T_2}^* = S_{T_2}^* \oplus R_{T_2}^* \), where \( S_{T_1}, S_{T_2} \) are unilateral shifts on \( S_{T_1} \) and \( S_{T_2} \), respectively and \( R_{T_1}, R_{T_2} \) are unilateral operator on \( R_{T_1} \) and \( R_{T_2} \), respectively. By Lemma 4.1, for each positive integer \( n \). There exist \( M_n \in \text{Lat}(T) \) and orthonormal set \( \{e_k^{(n)}\}_{k=1}^n \subset M_n \) such that

\begin{align*}
(6-a) & \quad e_k^{(n)} \in \ker(T_i | M_n)^*k \\
(6-b) & \quad [e_k^{(n)} \otimes e_k^{(n)}] = [C_0]_T,
\end{align*}

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$k = 1, \cdots, n, i = 1, 2$. Let $B_n = (B_{T_1|M_n}, B_{T_2|M_n}) \in \mathcal{L}(\tilde{K}_n)^{(2)}_{\text{comm}}$ be a minimal joint isometric dilation of $T|M_n = (T_1|M_n, T_2|M_n) = (T_1|M_n, T_2|M_n)$ obtained as $B_T|\tilde{K}_n$ for some $\tilde{K}_n \in \text{Lat}(B_T)$ and then $B^*_n = (B^*_{T_1|M_n}, B^*_{T_2|M_n})$ be the minimal joint coisometric extension of $(T|M_n)^* = (T_1|M_n^*, T_2|M_n^*)$.

Suppose

\[(6-c) \quad B^*_{T_1|M_n} = S^*_{T_1|M_n} \oplus R^*_{T_1|M_n} \text{ and } B^*_{T_2|M_n} = S^*_{T_2|M_n} \oplus R^*_{T_2|M_n}\]

where $S_{T_1|M_n}, S_{T_2|M_n}$ are unilateral shifts on $S_{T_1|M_n}, S_{T_2|M_n}$, respectively.

By (6-a) and (6-c), $e_k^{(n)} \in S_{T_i|M_n}, k = 1, \cdots, n, i = 1, 2$. Since $T$ has property $(H_1^{(2)})$, $S_{T_i|M_n} \subset S_{T_i}$ \forall $i$, then $e_k^{(n)} \in S_{T_i}$, for all pairs $k$ and $n$ with $i = 1, 2$. Hence we can prove

\[
\| [f_j \otimes w_1]T_i \| + \| [f_j \otimes w_2]T_i \| \to 0 \quad \forall w_i \in \mathcal{H}, \quad i = 1, 2.
\]

\[\square\]

REFERENCES


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