DUAL OPERATOR ALGEBRAS AND HEREDITARY PROPERTIES OF MINIMAL JOINT DILATIONS

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1. Introduction

Let $\mathcal{H}$ be a separable, infnite dimensional, complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. A *dual algebra* is a subalgebra of $\mathcal{L}(\mathcal{H})$ that contains the identity operator $I_{\mathcal{H}}$ and is closed in the ultraweak operator topology on $\mathcal{L}(\mathcal{H})$. The theory of dual algebras is deeply related to the study of the problem of solving systems of simultaneous equations in the predual of a dual algebra (see [1], [3] and [4]). In particular, Exner-Jung [11] defined certain Hereditary properties concerning a minimal isometric dilation of a contraction operator T and obtained some characterizations for membership in the the class $\mathbb{A}_{1,\aleph_0}$ which will be defined below. This give a motivation for this work.

Let T be a contraction operator in $\mathcal{L}(\mathcal{H})$ and let B_T be a minimal isometric dilation of T on $\mathcal{K}_+$,

$$\mathcal{K}_+ = \bigvee_{n=0}^{\infty} B_T^n \mathcal{H},$$

with the Wold decomposition $B_T = S_T \oplus R_T$, where $S_T \in \mathcal{L}(\mathcal{U}_T)$ is the unilateral shift part and $R_T \in \mathcal{L}(\mathcal{R}_T)$ is the residual part. Suppose that $T \in \mathcal{L}(\mathcal{H})$ has a non-zero semi-invariant subspace $\mathcal{M}$ (i.e, $\mathcal{M} \neq (0)$). For a compression $\widetilde{T} = T_{\mathcal{M}}$ of T to $\mathcal{M}$, we write a minimal isometric dilation of $\widetilde{T}$ by $B_{\widetilde{T}} = S_{\widetilde{T}} \oplus R_{\widetilde{T}}$. Recall that a contraction T has property (**H**) if, for any non-zero semi-invariant subspace $\mathcal{M}$ for T, the minimal isometric dilation $B_{T_{\mathcal{M}}} \in \mathcal{L}(\widetilde{\mathcal{K}})$ of $T_{\mathcal{M}}$ which is obtained as a restriction $B_T|\widetilde{\mathcal{K}}$ with $\widetilde{\mathcal{K}} \in \mathrm{Lat}(B_T)$ satisfies $\mathcal{U}_{T_{\mathcal{M}}} \subset \mathcal{U}_T$. In addition, a contraction operator $T \in \mathbb{A}$ has property ($\widetilde{\mathbf{P}}$) if there exists $\mathcal{M} \in \mathrm{Lat}(T)$ such that $T|\mathcal{M} \in \mathbb{A}(\mathcal{M})$ and

This work was partially supported by KOSEF 94-1400-02-01-3 and the Basic Science Research Institute Program, Ministry of Education, 1996, BSRI-94-1401.

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$T|\mathcal{M}$ has property $(\mathbf{P})$. Then it follows from [11] that $T \in \mathbb{A}_{1,\aleph_0}$ if and only if T has property $(\widetilde{\mathbf{P}})$. Also one discuss other related hereditary properties in [15]. In [18], one developed a functional calculus for a 2-tuple contractions. In [9], one introduced a class $\mathbb{A}_{m,n}^{(2)}$ of pairs of operators and obtained some results concerning minimal joint isometric dilations. So in this paper we extend the study of hereditary properties of single operators to 2-tuple operators.

In section 3 we will discuss certain Hereditary properties concerning minimal isometric dilations or minimal coisometric extensions of a pair of contractions. In section 3, we apply the hereditary properties to a theory of dual algebras.

2. Preliminaries

The notation and terminology employed here agree with those in [2], [4] and [19]. Suppose that $\mathcal{A}$ is a dual algebra in $\mathcal{L}(\mathcal{H})$. Let $\mathcal{C}_1 = \mathcal{C}_1(\mathcal{H})$ be the trace class in $\mathcal{L}(\mathcal{H})$ and let ${}^{\perp}\mathcal{A}$ denote the preannihilator of $\mathcal{A}$ in $\mathcal{C}_1$. Let $\mathcal{Q}_{\mathcal{A}}$ denote the quotient space $\mathcal{C}_1/{}^{\perp}\mathcal{A}$. One knows that $\mathcal{A}$ is the dual space of $\mathcal{Q}_{\mathcal{A}}$ and that the duality is given by

$$< T, [L] > = trace(TL), \; T \in \mathcal{A}, \; [L] \in \mathcal{Q}_{\mathcal{A}}.$$

For $T \in \mathcal{L}(\mathcal{H})$, let $\mathcal{A}_T$ denote the dual algebra generated by T. For vectors x and y in $\mathcal{H}$, we write, as usual, $x \otimes y$ for the rank one operator in $\mathcal{C}_1$ defined by $(x \otimes y)(u) = (u, y)x, \quad u \in \mathcal{H}$.

Throughout this paper, we write $\mathbb{N}$ for the set of natural numbers. We shall denote by $\mathbb{D}$ the open unit disc in the complex plane $\mathbb{C}$ and we write $\mathbb{T}$ for the boundary of $\mathbb{D}$.

For a Hilbert space $\mathcal{K}$ and any operators $T_i \in \mathcal{L}(\mathcal{K})$, $i = 1, 2$, we write $T_1 \cong T_2$ if T_1 is unitarily equivalent to T_2.

For $1 \leq p \leq \infty$ we denote the usual Lebesgue function space by $L^p = L^p(\mathbb{T})$ and the usual Hardy space by $H^p = H^p(\mathbb{T})$. One knows that the preannihilator ${}^{\perp}(H^\infty)$ of H^∞ in L^1 is the subspace H_0^1 consisting of those functions g in H^1 for which analytic extension $\widetilde{g}$ to $\mathbb{D}$ satisfies $\widetilde{g}(0) = 0$ (cf. [14]). It is well known that H^∞ is the dual space of L^1/H_0^1.

Suppose that m and n are any cardinal numbers such that $1 \leq m, n \leq \aleph_0$. A dual algebra $\mathcal{A}$ will be said to have property $(\mathbb{A}_{m,n})$ if every $m \times n$ system of simultaneous equations of the form $[x_i \otimes y_j] = [L_{ij}]$, $0 \leq i < m$, $0 \leq j < n$, where $\{[L_{ij}]\}_{\substack{0 \leq i < m \\ 0 \leq j < n}}$ is an arbitrary $m \times n$ array from $\mathcal{Q}_A$, has a solution $\{x_i\}_{0 \leq i < m}$, $\{y_j\}_{0 \leq j < n}$ consisting of a pair of sequences of vectors from $\mathcal{H}$. For brevity, we shall denote $(\mathbb{A}_{n,n})$ by $(\mathbb{A}_n)$. The class $\mathbb{A}(\mathcal{H})$ consists of all those absolutely continuous contractions T in $\mathcal{L}(\mathcal{H})$ for which the Foiaş-Nagy functional calculus $\Phi_T : H^\infty \longrightarrow \mathcal{A}_T$ is an isometry. Furthermore, we denote by $\mathbb{A}_{m,n}(\mathcal{H})$ the set of all T in $\mathbb{A}(\mathcal{H})$ such that the algebra $\mathcal{A}_T$ has property $(\mathbb{A}_{m,n})$. We write simply $\mathbb{A}_{m,n}$ for $\mathbb{A}_{m,n}(\mathcal{H})$ unless we mention otherwise.

Let $\mathcal{L}(\mathcal{H})^{(2)}_{comm}$ be the algebra of pairs of operators in $\mathcal{L}(\mathcal{H})$ which are commute. For $\mathbf{T} = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{comm}$, if there exists a 2-tuple $(S_1, S_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{comm}$ for some Hilbert space $\mathcal{K} \supset \mathcal{H}$ such that $\mathcal{H}$ is a semi-invariant subspace for S_j and $(S_j)_\mathcal{H} = T_j$, $j = 1, 2$. For $\mathbf{T} = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{comm}$, a joint dilation(extension) $\mathbf{S} = (S_1, S_2) \in \mathcal{L}(\mathcal{K})^{(2)}_{comm}$, where $\mathcal{K} \supset \mathcal{H}$, is said to be a joint unitary (resp. isometric, coisometric) dilation (extension) if each of S_j, $j = 1, 2$, is a unitary (resp. an isometry, a coisometry). If $\mathbf{U} = (U_1, U_2) \in \mathcal{L}(\mathcal{K})^{(2)}_{comm}$ is a joint dilation for $\mathbf{T} = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{comm}$ and $\mathcal{K}' \supset \mathcal{H}$ is a common invariant subspace for $\mathbf{U}$, then $\mathbf{U}|\mathcal{K}' = (U_1|\mathcal{K}', U_2|\mathcal{K}') \in \mathcal{L}(\mathcal{K}')^{(2)}_{comm}$ is a joint dilation of $\mathbf{T}$.

For $1 \leq p \leq \infty$, we denote by $L^p(\mathbb{T}^2)$ the Lebesgue spaces relative to normalized Lebesgue area measure on the torus $\mathbb{T}^2$ and by $H^p(\mathbb{T}^2)$ the Hardy space the subspaces of $L^p(\mathbb{T}^2)$ consisting of all the functions $f \in L^p(\mathbb{T}^2)$ such that the Foisson Kernel is analytic on $\mathbb{D}^2$. We write for $L^1_0(\mathbb{T}^2)$ the subspace of $L^1(\mathbb{T}^2)$ consisting of those functions $f \in L^1(\mathbb{T}^2)$ such that Fourier coefficient $^f(-n_1, -n_2) = 0$, for all $n_1, n_2 \in \mathbb{N}$.

The following provids a good relationship between $H^\infty(\mathbb{T}^2)$ and $\mathcal{A}_{T_1,T_2}$ which is a dual algebra generated by T_1, T_2.

Theorem 2.1. *[9, Theorem 2.4.2] If $(T_1, T_2) \in ACC^{(2)}(\mathcal{H})$, then there is an algebra homomorphism $\Phi_{T_1,T_2} : H^\infty(\mathbb{T}^2) \longrightarrow \mathcal{A}_{T_1,T_2}$ with the following properties:*

(a) $\Phi_{T_1,T_2}(1) = I_\mathcal{H}$, $\Phi_{T_1,T_2}(\omega_1) = T_1$, $\Phi_{T_1,T_2}(\omega_2) = T_2$, where ω_1 and ω_2 denote the coordinate functions.

(b) $\|\Phi_{T_1,T_2}(h)\| \leq \|h\|_\infty$, for all $h \in H^\infty(\mathbb{T}^2)$.

(c) Φ_{T_1,T_2} is weak-continuous.(i.e., continuous when both H^∞ and $\mathcal{A}_{T_1,T_2}$ are given the corresponding weak*-topologies).*

(d) The range of Φ_{T_1,T_2} is weak-dense in $\mathcal{A}_{T_1,T_2}$.*

(e) There is a bounded, linear, one-to-one map

$$\phi_{T_1,T_2} : \mathcal{Q}_{T_1,T_2} \longrightarrow L^1(\mathbb{T}^2)/L^1_0(\mathbb{T}^2)$$

with $\phi*_{T_1,T_2} = \Phi_{T_1,T_2}$.

(f) If Φ_{T_1,T_2} is an isometry, then it is a weak-homeomorphism onto $\mathcal{A}_{T_1,T_2}$ and ϕ_{T_1,T_2} is an isometry onto $L^1(\mathbb{T}^2)/L^1_0(\mathbb{T}^2)$.*

We now define the classes $\mathbb{A}^{(2)}(\mathcal{H})$ and $\mathbb{A}^{(2)}_{m,n}(\mathcal{H})$. The analogous classes $\mathbb{A}(\mathcal{H})$ and $\mathbb{A}_{m,n}(\mathcal{H})$ have been a central topic of study in the theory of dual algebras (cf. [4]). Let $(T_1,T_2) \in ACC^{(2)}(\mathcal{H})$. We say that $(T_1,T_2) \in \mathbb{A}^{(2)}(\mathcal{H})$ if the funtional calculus is an isometry. Furthermore, for $n \in \mathbb{N}$, we say that $(T_1,T_2) \in \mathbb{A}^{(2)}_{1/n}(\mathcal{H})$ if $(T_1,T_2) \in \mathbb{A}^{(2)}(\mathcal{H})$ and $\mathcal{A}_{T_1,T_2}$ has property $(\mathbb{A}_{1/n})$. Similarly, for m,n cardinal numbers with $1 \leq m, n\aleph_0$, we say that $\mathbb{A}^{(2)}_{m,n}(\mathcal{H})$ if $(T_1,T_2) \in \mathbb{A}^{(2)}(\mathcal{H})$ and $\mathcal{A}_{T_1,T_2}$ has property $(\mathbb{A}_{m,n})$. As before, we write $\mathbb{A}^{(2)}_n(\mathcal{H})$ instead of $\mathbb{A}^{(2)}_{n,n}(\mathcal{H})$.

Recall that $T \in C_{\cdot 0}$ if $\|T^{*n}x\| \longrightarrow 0$ for any $x \in \mathcal{H}$. We say $T \in C_{0\cdot}$ if $T^* \in C_{\cdot 0}$. And we denote that $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$.

3. Hereditary properties of joint dilations

Let $\mathbf{T} = (T_1,T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{comm}$ be a pair of contractions. A pair $(T_1,T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{comm}$ of contractions has a joint coisometric extension, and thus a minimal joint coisometric extension(cf.[,]). Let$\mathbf{B_T} = (B_1,B_2) \in \mathcal{L}(\mathcal{K})^{(2)}_{comm}$ be a minimal joint isometric dilation of $T = (T_1,T_2)$, so that $\mathcal{K} \supset \mathcal{H}$, $\mathcal{H}$ is a common invariant subspace for B_j, and $B_j^*|\mathcal{H} = T_j^*$, $j = 1,2$. Then $\mathcal{K}$ has decomposition $\mathcal{K} = \mathcal{S}_j \oplus \mathcal{R}_j$, $j = 1,2$, such that $\mathcal{S}_j$, $\mathcal{R}_j$ are reducing subspaces for B_j $j = 1,2$ respectively and $B_j|\mathcal{S}_j = S_j^*$, and $B_j|\mathcal{R}_j = R_j$, $j = 1,2$, where S_j^* is backward shifts operator of some multiplicity and R_j is unitary operator $j = 1,2$. Furthermore, it follows that $\mathbf{B_{T^*}} = (B_1^*,B_2^*) \in \mathcal{L}(\mathcal{K})^{(2)}_{comm}$ is a minimal joint coisometric extension of $\mathbf{T}^* = (T_1^*,T_2^*)$.

Let $\mathcal{M}$ be a common invariant subspace for $\mathbf{T} = (T_1,T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{comm}$ with $\mathcal{M} \neq (0)$. Then a minimal joint isometric dilation $\mathbf{B_T} = (B_1,B_2)$ of $\mathbf{T}$ is a joint isometric dilation of $\mathbf{T}|\mathcal{M} = (T_1|\mathcal{M}, T_2|\mathcal{M})$. Hence $\mathbf{T}|\mathcal{M}$ has a minimal isometric dilation $\mathbf{B_{T|\mathcal{M}}} \in \mathcal{L}(\widetilde{\mathcal{K}})^{(2)}_{comm}$ with $\mathbf{B_{T|\mathcal{M}}} = (B_1|\widetilde{\mathcal{K}}, B_2|\widetilde{\mathcal{K}})$ such that $\mathcal{M} \subset \widetilde{\mathcal{K}} \subset \mathcal{K}$ with $\widetilde{\mathcal{K}} \in \mathrm{Lat}(\mathbf{B_T})$ and $\mathbf{B}_{T|\mathcal{M}} = \mathbf{B}_T|\widetilde{\mathcal{K}}$. Note that contrary to the one variable case, all minimal unitary dilation of a pair of contractions are not isometric. Hence we should define hereditary properties with a slight diferrence with one variable case as follows.

Definition 3.1. Suppose that $\mathbf{T} = (T_1,T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{comm}$ be a pair of contractions.

(1) $\mathbf{T}$ has property $(\mathbf{H}_1^{(2)})$ if there exists minimal joint isometric dilation $\mathbf{B_T}$ of $\mathbf{T}$ such that for any non-zero common invariant subspace $\mathcal{M}$ for

$\mathbf{T} = (T_1, T_2)$ the minimal joint isometric dilation $\mathbf{B}_{\mathbf{T}|\mathcal{M}} = (B_1|\widetilde{\mathcal{K}}, B_2|\widetilde{\mathcal{K}}) \in \mathcal{L}(\widetilde{\mathcal{K}})^{(2)}_{comm}$ of $\mathbf{T}|\mathcal{M}$ obtained as a restriction $\mathbf{B}_{\mathbf{T}}|\widetilde{\mathcal{K}}$ with $\widetilde{\mathcal{K}} \in$ Lat$(\mathbf{B}_{\mathbf{T}})$ satisfies $\mathcal{S}_{T_1|\mathcal{M}} \subset \mathcal{S}_{T_1}$ and $\mathcal{S}_{T_2|\mathcal{M}} \subset \mathcal{S}_{T_2}$.

(2) $\mathbf{T}$ has property $(\mathbf{H}_1^{(2)*})$ if there exists minimal joint coisometric extision $\mathbf{B}_{\mathbf{T}}$ of $\mathbf{T}$ such that for any non-zero common invariant subspace $\mathcal{M}$ for $\mathbf{T} = (T_1, T_2)$, the minimal joint coisometric extension $\mathbf{B}'_{\mathbf{T}|\mathcal{M}} = (B'_1|\widetilde{\mathcal{K}}, B'_2|\widetilde{\mathcal{K}}) \in \mathcal{L}(\widetilde{\mathcal{K}})^{(2)}_{comm}$ of $\mathbf{T}|\mathcal{M}$ obtained as a restriction $\mathbf{B}'_{\mathbf{T}}|\widetilde{\mathcal{K}}$ with $\widetilde{\mathcal{K}} \in$ Lat$(\mathbf{B}'_{\mathbf{T}})$ satisfies $\mathcal{S}_{T_1|\mathcal{M}} \subset \mathcal{S}_{T_1}$ and $\mathcal{S}_{T_2|\mathcal{M}} \subset \mathcal{S}_{T_2}$.

(3) $\mathbf{T}$ has property $(\mathbf{H}_2^{(2)})$ if there exists minimal joint isometric dilation $\mathbf{B}_{\mathbf{T}}$ of $\mathbf{T}$ such that for any non-zero common invariant subspace $\mathcal{M}$ for $\mathbf{T} = (T_1, T_2)$ the minimal joint isometric dilation $\mathbf{B}_{\mathbf{T}|\mathcal{M}} = (B_1|\mathcal{M}, B_2|\mathcal{M}) \in \mathcal{L}(\widetilde{\mathcal{K}})^{(2)}_{comm}$ of $\mathbf{T}|\mathcal{M}$ obtained as a restriction $\mathbf{B}_{\mathbf{T}}|\widetilde{\mathcal{K}}$ with $\widetilde{\mathcal{K}} \in$ Lat$(\mathbf{B}_{\mathbf{T}})$ satisfies $\mathcal{R}_{T_1|\mathcal{M}} \subset \mathcal{R}_{T_1}$ and $\mathcal{R}_{T_2|\mathcal{M}} \subset \mathcal{R}_{T_2}$.

(4) $\mathbf{T}$ has property $(\mathbf{H}_2^{(2)*})$ if there exists minimal joint coisometric extension $\mathbf{B}_{\mathbf{T}}$ of $\mathbf{T}$ such that for any non-zero common invariant subspace $\mathcal{M}$ for $\mathbf{T} = (T_1, T_2)$, the minimal joint coisometric extension $\mathbf{B}'_{\mathbf{T}|\mathcal{M}} = (B'_1|\widetilde{\mathcal{K}}, B'_2|\widetilde{\mathcal{K}}) \in \mathcal{L}(\widetilde{\mathcal{K}})^{(2)}_{comm}$ of $\mathbf{T}|\mathcal{M}$ obtained as a restriction $\mathbf{B}'_{\mathbf{T}}|\widetilde{\mathcal{K}}$ with $\widetilde{\mathcal{K}} \in$ Lat$(\mathbf{B}'_{\mathbf{T}})$ satisfies $\mathcal{R}_{T_1|\mathcal{M}} \subset \mathcal{R}_{T_1}$ and $\mathcal{R}_{T_2|\mathcal{M}} \subset \mathcal{R}_{T_2}$.

We denote $C_{\cdot 0}^{(2)}$ is the set of pairs (T_1, T_2) of operators on $\mathcal{H}$ with $T_i \in C_{\cdot 0}(\mathcal{H})$ $i = 1, 2$.

Proposition 3.2. *Suppose that* $\mathbf{T} = (T_1, T_2) \in C_{\cdot 0}^{(2)}(\mathcal{H})$. *Then* $\mathbf{T}$ *has property* $(\mathbf{H}_1^{(2)})$.

Proof. Since $T_i \in C_{\cdot 0}$, B_{T_i} is a unilateral shift on $\mathcal{H}_i$. Let us consider a joint isometric dilation $\mathbf{B} := (B_{T_1} \oplus B_{T_2}, B_{T_1} \oplus B_{T_2})$ in $\mathcal{L}(\mathcal{H}_1 \oplus \mathcal{H}_2)^{(2)}_{comm}$ of $\mathbf{T}$.

Then $\mathbf{B} \in C_{\cdot 0}^{(2)}$ and there exists a minimal joint isometric dilation $\mathbf{B}_{\mathbf{T}}$ of $\mathbf{T}$ such that $\mathbf{B}|\mathcal{K} = \mathbf{B}_T := (B'_{T_1}, B'_{T_2})$. Since $\mathbf{B} \in C_{\cdot 0}^{(2)}$, $\mathbf{B}_{\mathbf{T}} \in C_{\cdot 0}^{(2)}$. Consider $\mathcal{M} \in$ Lat$(\mathbf{T})$ with $\mathbf{T}|\mathcal{M} := (T_1|\mathcal{M}, T_2|\mathcal{M}) \in C_{\cdot 0}^{(2)}$. Then since there is no unitary part, $\mathbf{T}$ has property $(\mathbf{H}_1^{(2)})$. □

Example 3.3. Let U be a bilateral shift and let consider $\mathbf{U}$=(U, U). Then $\mathbf{U}$=$\mathbf{B}_{\mathbf{U}}$ and $\mathcal{S} = (0)$. Let $\mathcal{M} \in$ Lat$(\mathbf{U})$. Then $\mathbf{U}|\mathcal{M} = \mathbf{B}_{\mathbf{U}}|\mathcal{M}$ and $\widetilde{\mathcal{S}} = \mathcal{M}$. Hence $\widetilde{\mathcal{S}}$ can not be contained in $\mathcal{S}$ and $\mathbf{U}$ does not have property $(\mathbf{H}_1^{(2)})$.

Theorem 3.4. *Every pair of contractions* $\mathbf{T} = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{comm}$ *has*

(1) *property* $(\mathbf{H}_1^{(2)*})$,

(2) *property* $(\mathbf{H}_2^{(2)})$,

(3) *property* $(\mathbf{H}_2^{(2)*})$.

Proof. Let $\mathbf{T} = (T_1, T_2) \in \mathcal{L}(\mathcal{H})^{(2)}_{comm}$ and $\mathcal{M}$ be a nontrivial comman invariant subspace of $\mathbf{T}$.

(1) Let $\mathbf{B}'_{\mathbf{T}} = (B'_{T_1}, B'_{T_2})$, be the Ando coismetric extension of $\mathbf{T}$ and let $\mathbf{B}'_{\widetilde{\mathbf{T}}} = (B'_{\widetilde{T}_1}, B'_{\widetilde{T}_2})$ be a minimal joint coisometric extension of $\widetilde{\mathbf{T}} = (\widetilde{T}_1, \widetilde{T}_2)$ such that $\widetilde{\mathcal{K}} \subset \mathcal{K}$. Then

$$B'_{T_i} = S^*_{T_i} \oplus R^*_{T_i} \in \mathcal{L}(\mathcal{S}_{T_i} \oplus \mathcal{R}_{T_i}), i = 1, 2$$

and

$$\begin{aligned} B'_{\widetilde{T}_i} &= S^*_{\widetilde{T}_i} \oplus R^*_{\widetilde{T}_i} \in \mathcal{L}(\mathcal{S}_{\widetilde{T}_i} \oplus \mathcal{R}_{\widetilde{T}_i}), \qquad i = 1, 2 \\ &\cong \begin{pmatrix} T_i|\mathcal{M} & * \\ 0 & * \end{pmatrix} \end{aligned}$$

relative to a decomposition $\mathcal{M} \oplus (\widetilde{\mathcal{K}} \ominus \mathcal{M})$.

We now claim that $\mathcal{S}_{\widetilde{T}_i} \subset \mathcal{S}_{T_i}$, $i = 1, 2$. Let $x \in \mathcal{S}_{\widetilde{T}_i}$ and let $x = s \oplus r \in \mathcal{S}_{T_i} \oplus \mathcal{R}_{T_i}$. Since $B'_{\widetilde{T}_i} = B'_{T_i}|\widetilde{\mathcal{K}}$, by minimality where $\widetilde{\mathcal{K}} \in \mathrm{Lat}(B'_{T_i})$, we have

$$\begin{aligned} \|S^{*n}_{\widetilde{T}_i} x\|^2 = \|B'^n_{\widetilde{T}_i} x\|^2 &= \|B'^n_{T_i} x\|^2 \\ &= \|(S^{*n}_{T_i} \oplus R^{*n}_{T_i})(s \oplus r)\|^2 \\ &= \|S^{*n}_{T_i} s\|^2 + \|R^{*n}_{T_i} r\|^2 \\ &= \|S^{*n}_{T_i} s\|^2 + \|r\|^2 \end{aligned}$$

let $n \to \infty$, $r = 0$. Therefore $x \in \mathcal{S}_{T_i}$, $i = 1, 2$. Hence $\mathbf{T}$ has property $(\mathbf{H}_1^{(2)*})$.

(3) Using notation in the proof of $(\mathbf{H}_1^*.)$. We claim that $\mathcal{R}_{\widetilde{T}_1} \subset \mathcal{R}_{T_1}$. Let $x \in \mathcal{R}_{\widetilde{T}_1} \subset \mathcal{K} = \mathcal{S}_{T_1} \oplus \mathcal{R}_{T_1}$ and let $x = s \oplus r \in \mathcal{S}_{T_1} \oplus \mathcal{R}_{T_1}$. Then we have

$$\begin{aligned} \|s\|^2 + \|r\|^2 = \|x\|^2 = \|\mathcal{R}^{*n}_{\widetilde{T}_1} x\|^2 &= \|B'^n_{\widetilde{T}_1} x\|^2 \\ = \|B'^n_{T_1} x\|^2 &= \|S^{*n}_{T_1} s\|^2 + \|R^{*n}_{T_1} r\|^2 \\ &= \|S^{*n}_{T_1} s\|^2 + \|r\|^2. \end{aligned}$$

Let $n \to \infty$, since $\|S_{T_1}^{*n}s\| \to 0$, $s = 0$. Therefore $\mathcal{R}_{\widetilde{T}_1} \subset \mathcal{R}_{T_1}$. Similarly $\mathcal{R}_{\widetilde{T}_2} \subset \mathcal{R}_{T_2}$. Hence $\mathbf{T}$ has property $(\mathbf{H}_2^{(2)*})$.

(2) Let $\mathbf{B_T} \in \mathcal{L}(\mathcal{K})^{(2)}_{comm}$ be the Ando minimal joint isometric dilation of $\mathbb{T}$ and $\mathbf{B}_{\widetilde{\mathbf{T}}} = \mathbf{B}_{\mathbf{T}|\mathcal{M}} \in \mathcal{L}(\widetilde{\mathcal{K}})^{(2)}_{comm}$ be the minimal joint isometric dilation of and $\widetilde{\mathbf{T}} = \mathbf{T}|\mathcal{M} = (T_1|\mathcal{M}, T_2|\mathcal{M})$ such that $\mathbf{B_T}|\widetilde{\mathcal{K}} = \mathbf{B}_{\widetilde{\mathbf{T}}}$ and $\widetilde{\mathcal{K}} \in \mathrm{Lat}(B_1, B_2)$. We have $B_{T_i} = S_{T_i} \oplus R_{T_i} \in \mathcal{L}(\mathcal{S}_{T_i} \oplus \mathcal{R}_{T_i})$ and $B_{\widetilde{T}_i} = S_{\widetilde{T}_i} \oplus R_{\widetilde{T}_i} \in \mathcal{L}(\mathcal{S}_{\widetilde{T}_i} \oplus \mathcal{R}_{\widetilde{T}_i})$. We shall claim $\mathcal{R}_{\widetilde{T}_i} \subset \mathcal{R}_{T_i}, i = 1, 2$. Let $x \in \mathcal{R}_{\widetilde{T}_1}$ and let $x = s \oplus r \in \mathcal{S}_{T_1} \oplus \mathcal{R}_{T_1}$. Since $B_{\widetilde{T}_1} = B_{T_1}|\widetilde{\mathcal{K}}$,

$$B_{T_1}^{*n} = \begin{pmatrix} B_{\widetilde{T}_1}^{*n} & 0 \\ A_n & * \end{pmatrix}$$

relative to a decomposition $\widetilde{\mathcal{K}} \oplus (\mathcal{K} \ominus \widetilde{\mathcal{K}})$ for any $n \in \mathbb{N}$.

$$\begin{aligned} \|x\|^2 &\le \|x\|^2 + \|A_n x\|^2 = \|R_{\widetilde{T}_1}^{*n} x\|^2 + \|A_n x\|^2 \\ &= \|R_{\widetilde{T}_1}^{*n} x \oplus A_n x\|^2 = \|B_{T_1}^{*n} x\|^2 \\ &\le \|x\|^2. \end{aligned}$$

Therefore $\|x\| = \|B_{T_1}^{*n} x\|$ for any $n \in \mathbb{N}$. And we have

$$\begin{aligned} \|s\|^2 + \|r\|^2 &= \|x\|^2 \\ &= \|B_{T_1}^{*n} x\|^2 = \|S_{T_1}^{*n} s\|^2 + \|R_{T_1}^{*n} r\|^2 \\ &= \|S_{T_1}^{*n} s\|^2 + \|r\|^2. \end{aligned}$$

Letting $n \to \infty$, $\|S_{T_1}^{*n}s\| \to 0$, $s = 0$. Therefore $x \in \mathcal{R}_{T_1}$. So $\mathcal{R}_{\widetilde{T}_1} \subset \mathcal{R}_{T_1}$. Similarly $\mathcal{R}_{\widetilde{T}_2} \subset \mathcal{R}_{T_2}$. Hence $\mathbf{T}$ has property $(\mathbf{H}_2^{(2)})$. □

4. Dual algebras generated by 2-tuple contractions

Lemma 4.1. *If $\mathbf{T} = (T_1, T_2) \in \mathbf{A}_{1,1}^{(2)}(\mathcal{H})$, then for any positive integer n, there exists $\mathcal{M}_n \in Lat(\mathbf{T})$ and $\{e_k^{(n)}\}_{k=1}^n \subset \mathcal{M}_n$ such that*

$$e_k^{(n)} \in \; Ker(T_i|\mathcal{M}_n)^{*k} \ominus \; Ker(T_i|\mathcal{M}_n)^{*k-1}$$

and

$$[e_k^{(n)} \otimes e_k^{(n)}] = [C_0]_{\mathbf{T}}$$

$k = 1, \cdots n$.

Proof. Let us consider the operator $A_j \in \mathcal{L}(\mathbf{C}^n)$, $j = 1, 2$ such that

$$A_j = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Then it is easy to show that A_j has a cyclic vector By [17. Theorem 3.3.1], there exist $\mathcal{M}_n, \mathcal{N}_n \in \text{Lat}(\mathbf{T})$ with $\mathcal{M}_n \supset \mathcal{N}_n$ $\dim(\mathcal{M}_n \ominus \mathcal{N}_n) = n$ such that $A_i' = T_{i\mathcal{M}_n} \ominus \mathcal{N}_n$ is similar to A_i. Let $X : \mathcal{M}_n \ominus \mathcal{N}_n \longrightarrow \mathcal{H}$ invertible operator with $A_i = X A_i' X^{-1}$ for $i = 1, 2$, then $XA_i' = A_i X$ and then $X^* A_i^* = A_i'^* X^*$. Define $e_k^{(n)} = X^* u_k$,where u_k. Then

$$(A_i'^*)^j e_k^{(n)} = A_i'^{*j} X^* u_k = X^* A_i^{*j} u_k.$$

If $j \leq k$, then $X^* A_i^{*j} u_k = 0$. Since X^* is one to one, if $j \leq k$, then $X^* A_i^{*j} u_k \neq 0$, for $i = 1, 2$. Hence we have

$$e_k^{(n)} \in \text{Ker}(A_i'^{*k})^{*k} \ominus \text{Ker}(A_i'^{*k})^{*k-1}.$$

Moreover, since

$$(T_i|\mathcal{M}_n)^* = \begin{pmatrix} * & 0 \\ * & A_i'^{*k} \end{pmatrix},$$

we have

$$e_k^{(n)} \in \text{Ker}(T_i|\mathcal{M}_n^*)^k \ominus \text{Ker}(T_i|\mathcal{M}_n^*)^{k-1}.$$

Thus, for $h \in H^\infty(\mathbf{T})$,

$$h(T_1, T_2) = \begin{pmatrix} * & * & * \\ 0 & D & * \\ 0 & 0 & * \end{pmatrix},$$

where D is the scalar operater(with respect to basis $\{u_n\}$) defined by $De_k^{(n)} = h(\lambda)e_k^{(n)}$, therefore

$$< h(T_1, T_2)e_k^{(n)}, e_k^{(n)} >= h(\lambda).$$

Then by definition of

$$[C_\lambda]_{\mathbf{T}} = [e_k^{(n)} \otimes e_k^{(n)}]_{\mathbf{T}}, k = 1, \cdots n.$$

Hence the proof is complete. □

Let $\mathcal{A}$ be a dual algebra, for $0 \le \theta < \gamma \le 1$. We recall from [ ] that $\mathcal{E}_\theta^r(\mathcal{A})$ is the set of all $[L]$ in $\mathcal{Q}_{\mathcal{A}}$ such that there exist sequences $\{x_i\}$ and $\{y_i\}$ in the unit ball of $\mathcal{H}$ satisfying

$$\limsup_{i \to \infty} \|[L] - [x_i \otimes y_i]\| \le \theta$$

and

$$\|[x_i \otimes z]\| \longrightarrow 0, \qquad \forall z \in \mathcal{H}.$$

Furthermore, for some $0 \le \theta < \gamma \le 1$, $\mathcal{A}$ has property $E_{\theta,\gamma}^r$ if $\overline{aco}(\mathcal{E}_\theta^r(\mathcal{A})) \supset B_{0,\gamma} = \{[L] \in Q : \|[L]\| \le \gamma\}$.

Theorem 4.2. *Suppose $\mathbf{T} = (T_1, T_2) \in \mathbf{A}_{1,1}^{(2)}(\mathcal{H})$ has property $(\mathbf{H}_1^{(2)})$. Then there exist $\{f_j\}_{j=1}^\infty$ of unit vectors in $\mathcal{H}$ satisfying*

$$[f_j \otimes f_j] = [C_0]_{\mathbf{T}}$$

and

$$\|[f_j \otimes w_1]_{T_1}\| + \|[f_j \otimes w_2]_{T_2}\| \longrightarrow 0 \qquad \forall w_i \in \mathcal{H}, \qquad i = 1, 2.$$

Proof. Let $\mathbf{B_T} = (B_{T_1}, B_{T_2})$ be a minimal joint isometric dilation of $\mathbf{T} = (T_1, T_2)$ has property $(\mathbf{H}_1^{(2)})$, and let $\mathbf{B}_{\mathbf{T}}^* = (B_{T_1}^*, B_{T_2}^*)$ be a minimal joint co-isometric extension of $T^* = (T_1^*, T_2^*)$. Suppose $B_{T_1}^* = S_{T_1}^* \oplus R_{T_1}^*$, $B_{T_2}^* = S_{T_2}^* \oplus R_{T_2}^*$, where S_{T_1}, S_{T_2} are unilateral shifts on $\mathcal{S}_{T_1}$ and $\mathcal{S}_{T_2}$, respectively and R_{T_1}, R_{T_2} are unilateral operator on $\mathcal{R}_{T_1}$ and $\mathcal{R}_{T_2}$, respectively. By Lemma 4.1, for each positive integer n. There exist $\mathcal{M}_n \in \mathrm{Lat}(\mathbf{T})$ and orthonormal set $\{e_k^{(n)}\}_{k=1}^n \subset \mathcal{M}_n$ such that

(6-a) $$e_k^{(n)} \in \ \mathrm{Ker}(T_i|\mathcal{M}_n)^{*k}$$

and

(6-b) $$[e_k^{(n)} \otimes e_k^{(n)}] = [C_0]_{\mathbf{T}},$$

$k = 1, \cdots n$, $i = 1, 2$. Let $\mathbf{B}_n = (B_{T_1|\mathcal{M}_n}, B_{T_2|\mathcal{M}_n}) \in \mathcal{L}(\widetilde{\mathcal{K}}_n)^{(2)}_{comm}$ be a minimal joint isometric dilation of $\mathbf{T}|\mathcal{M}_n = (T_1|\mathcal{M}_n, T_2|\mathcal{M}_n) = (T_1|\mathcal{M}_n, T_2|\mathcal{M}_n)$ obtained as $\mathbf{B_T}|\widetilde{\mathcal{K}}_n$ for some $\widetilde{\mathcal{K}}_n \in \mathrm{Lat}(\mathbf{B_T})$ and then $\mathbf{B}_n^* = (B^*_{T_1|\mathcal{M}_n}, B^*_{(T_2)|\mathcal{M}_n})$ be the minimal joint coisometric extension of

$$(\mathbf{T}|\mathcal{M}_n)^* = (T_1|\mathcal{M}_n^*, T_2|\mathcal{M}_n^*).$$

Suppose

$$\text{(6-c)} \qquad B^*_{T_1|\mathcal{M}_n} = S^*_{T_1|\mathcal{M}_n} \oplus R^*_{T_1|\mathcal{M}_n} \text{and} B^*_{T_2|\mathcal{M}_n} = S^*_{T_2|\mathcal{M}_n} \oplus R^*_{T_2|\mathcal{M}_n}$$

where $S_{T_1|\mathcal{M}_n}, S_{T_2|\mathcal{M}_n}$ are unilateral shifts on $\mathcal{S}_{T_1|\mathcal{M}_n}, \mathcal{S}_{T_2|\mathcal{M}_n}$, respectively. By (6-a) and (6-c), $e_k^{(n)} \in \mathcal{S}_{T_i|\mathcal{M}_n}$, $k = 1, \cdots n$, $i = 1, 2$. Since T has property $(\mathbf{H}_1^{(2)})$, $\mathcal{S}_{T_i|\mathcal{M}_n} \subset \mathcal{S}_{T_i}$ $\forall n$ $\forall i$, then $e_k^{(n)} \in \mathcal{S}_{T_i}$, for all pairs k and n with $i = 1, 2$. Hence we can prove

$$\|[f_j \otimes w_1]_{T_1}\| + \|[f_j \otimes w_2]_{T_2}\| \longrightarrow 0 \qquad \forall w_i \in \mathcal{H}, \qquad i = 1, 2.$$

□

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