Galois Groups of Subfactors

Jeong-Hee Hong

Abstract

We review the subfactor theory in terms of bimodules and describe the Galois group of subfactors through 1 dimensional bimodules appearing in some level of derived tower. Some examples of the Galois groups are given in the case of fixed point algebras, crossed product algebras, and their compositions.

1 Introduction

After A. Ocneanu ([13]) has introduced a paragroup as an invariant for the classification of subfactors, the study of bimodules plays an important role in the theory of subfactors, initiated by V. Jones ([7]).

The paragroup for \( N \subset M \) with finite Jones index is a grouplike object which provides information about the position of the subfactor \( N \) in a type \( II_1 \) factor \( M \). Indeed, a paragroup consists of a principal graph and a family of anti-automorphisms, which requires the study of bimodules and their intertwiners.

The first part of the paper is devoted to give an exposition on elementary theory of subfactors in terms of bimodules. Based on this notion, we show how the vertices of the principal graph is related to certain irreducible bimodules. Then we clarify the Galois groups for the inclusion by counting on the dimensions of intertwiners spaces in Section 3. In particular, we examine the result with various examples, like fixed point algebras, crossed product algebras, and their compositions.

It is worth to mention that there are several equivalent ways of describing paragroups, for example, Longo's sector theory ([11]) which is an analogous
to bimodule theory. While properly infinite factor cases require sector theory naturally, we use a bimodule approach for the type $II_1$ factor cases. Among various materials about bimodules (sectors) associated to a subfactor, we refer [5, 8, 9, 10, 11, 15] for the references.

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2 Preliminaries

2.1 Bimodules

Let $A$ and $B$ be type $II_1$ factors. When $B^\circ$ denotes the opposite algebra of $B$, an $A$-$B$ bimodule $X = _A X_B$ is defined as a Hilbert space $X$ with normal $*$-representations of $A$ and $B^\circ$ which commute each other, equipped with actions $a \cdot \xi \cdot b$, where $a \in A, b \in B$ and $\xi \in X$.

We say that two $A$-$B$ bimodules $X$ and $Y$ are said to be unitary equivalent, denoted by $X \cong Y$, if there exists a surjective isometry $u : X \to Y$ commuting with the left and right actions, i.e. $u(a \cdot \xi \cdot b) = a \cdot u(\xi) \cdot b$. The categorical operations on the set of bimodules are given as follows.

1. The conjugate (contragredient) $B$-$A$ bimodule $BX^*_A$ of $_A X_B$ is the conjugate Hilbert space $X^*$, with the action $b \cdot \xi^* \cdot a = (a^* \cdot \xi \cdot b^*)^*$.

2. Let $C$ be a type $II_1$ factor. Then the relative tensor product of $A$-$B$ bimodule $_A X_B$ and $B$-$C$ bimodule $BY_C$ over $B$ is given by the $A$-$C$ bimodule $AX \otimes_B Y_C$.

We have $(X^*)^* \cong X$ and $(AX \otimes_B Y_C)^* \cong CY^* \otimes_B X^* A$. An $A$-$A$ bimodule $X$ is called self-conjugate if $_A X^*_A \cong AX_A$.

The statistical dimension of the bimodule of $X$ is defined as

$$d(X) = d(_A X_B) = \sqrt{dim(AX) \cdot dim(X_B)}.$$ 

A bimodule $X$ is said to have finite type if $d(X) < \infty$. It is clear that $d(X) \geq 0$ for nonzero bimodule $X$. Moreover, the dimension function $d$
on finite type bimodules satisfies algebraic properties; \( d(X^*) = d(X) \), \( d(X \oplus Y) = d(X) + d(Y) \) and \( d(X \otimes Y) = d(X)d(Y) \). Also \( d \) is a conjugacy invariant function, i.e. when \( X \cong Y \), we have \( d(X) = d(Y) \).

Let \( \text{Hom}(X, Y) \) be the set of intertwiners from \( \text{A-B bimodule } X =_A X_B \) to \( Y =_A Y_B \) which commute with right and left actions of \( A \) and \( B \) resp., that is, a set of bounded linear operator \( T : X \to Y \) satisfying \( T(a \cdot \xi \cdot b) = a \cdot T(\xi) \cdot b \). When \( X = Y \), we denote by \( \text{End}(X) \). Then \( \text{End}(X) = \text{End}_B(A X_B) \cong A' \cap \text{End}(X) \cap (B^\omega)' \), which is a von Neumann algebra on \( X \). A bimodule \( X \) is called irreducible if \( \text{End}(X) = CI_X \). When \( X \) is an \( \text{A-B bimodule of finite type} \), \( X \) can be decomposed into finitely many direct sum of irreducible submodules as follows;

\[ A X_B \cong \sum \oplus m_i X_i \text{ with irreducible } \text{A-B bimodules } X_i, \]

where \( m_i = \dim \text{Hom}(X, X_i) \neq 0 \). Note that \( X_i \cong pX \) for a minimal projection \( p \) on \( X \).

We end this section with the following useful observation ([10, 15]). Let \( A \) be a type \( II_1 \) factor and \( L^2(A) \) the Hilbert space by GNS completion of \( A \) with respect to the inner product defined by the trace. Then the normal representations of \( A \) and \( A^\omega \) play the left and right actions which provide the bimodule structure to its Hilbert space \( \text{AL}(A)_A \) with actions

\[ a \cdot \xi \cdot b = a J_A b^* J_A \xi \text{ for } a, b \in A, \xi \in L^2(A), \]

where \( J_A \) denotes the modular convolution of \( L^2(A) \).

**Lemma 2.1** Let \( A \) and \( B \) be type \( II_1 \) factors and \( A X_B \) and \( A Y_B \) the \( \text{A-B bimodules of finite type} \). Then the followings hold;

1. \( \text{Hom}(X, Y) \cong \text{Hom}(Y^*, X^*) \)

2. \( \text{Hom}(X, Y) \cong \text{Hom}(\text{AL}(A)_A, Y \otimes_B X^*) \)

**Theorem 2.2** (Frobenius reciprocity) Let \( A, B \) and \( C \) be type \( II_1 \) factors and \( A X_{B,B} Y_{C,A} Z_C \) the bimodules of finite types. Then the following vector spaces are isomorphic;

\[ \text{Hom}(A X \otimes_B Y_{C,A} Z_C) \cong \text{Hom}(A Z \otimes C Y_{B,A} X_B) \cong \text{Hom}(B X^* \otimes_A Z_{C,B} Y_C). \]

**Proof.** It follows from Lemma 2.1, using the contragredient bimodules. \( \square \)

We often use this fact in the sense that they have the same dimension over \( C \).
2.2 Bimodules associated to $N \subset M$

Let $M$ be a type $II_1$ factor. Let $L^2(M)$ be the Hilbert space by GNS completion of $M$ with respect to the inner product defined by the trace. The normal representations of $M$ and $M^\circ$ play the left and right actions which provide the bimodule structure to its Hilbert space $M L^2(M)_M$ with actions

$$n \cdot \xi \cdot m = n J_M m^* J_M \xi , \text{ for } n, m \in M, \xi \in L^2(M),$$

where $J_M$ denotes the modular convolution of $L^2(M)$.

For a given inclusion of factors $N \subset M$, we now associate a $N$-$M$ bimodule. By restricting left action to $N$, the associated bimodule is defined by $H \triangleq N H_M = N L^2(M)_M$, which becomes a $N$-$M$ bimodule with the actions $n \cdot \xi \cdot m$ ($n \in N, m \in M$). Then conjugate $M$-$N$ bimodule of $H$ is given by $H^* = M L^2(M)_N \cong M L^2(M)_N = M H_N$, i.e. the same Hilbert space $L^2(M)$ with restricted right action to $N$.

The Jones’s index of $N \subset M$ is defined by $[M : N] = dim_N H$, the coupling constant of $N$ on the Hilbert space $L^2(M)$. When $[M : N] < \infty$, the statistical dimension of the $N$-$M$ bimodule $H$ is

$$d(H) = d(N H_M) = \sqrt{dim_N H \dim H_M} = [M : N]^{\frac{1}{2}}.$$

Hence $N$-$M$ bimodule $H$ is of finite type when $[M : N]$ is finite. Moreover the fact of $End(H) \cong B(L^2(H))$ gives that

$$End(N H_M) = N' \cap (M^\circ)' \cap End(H) = N' \cap M,$$

and so $H$ is an irreducible $M$-$N$ bimodule if and only if $N$ is an irreducible subfactor of $M$.

2.3 The graph invariants

For a pair of factors $N \subset M$, the basic construction $M_1$ for $N \subset M$ is defined as $M_1 = End(H_N)$, the algebra of endomorphisms of $M$ viewed as a right $N$-module. Indeed,

$$M_1 = (N^\circ)' \cap End(H) = J_M N' J_M \supset J_M M' J_M = M.$$
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When \([M : N] < \infty\), \(M_1\) is also type \(II_1\) factor with \([M_1 : M] = [M : N]\). This fact enables us to iterate the basic constructions to obtain Jones tower of \(II_1\) factors;

\[N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots.\]

The derived tower is an increasing sequence of relative commutants algebras

\[C_1 = N' \cap N \subset N' \cap M \subset N' \cap M_1 \subset N' \cap M_2 \subset \cdots.\]

The following observation provides the bimodule description of the derived tower. Note that \(H \otimes H^* = L^2(M) \otimes_M L^2(M) =_N L^2(M)_N\), \(N\)-\(N\) bimodule with left and right actions restricted to \(N\). Due to the fact \(M_1 = L^2(M) \otimes_N L^2(M)\), we also see that

\[H \otimes H^* \otimes H =_N L^2(M) \otimes N L^2(M) \otimes_M L^2(M) =_N L^2(M_1)_M.\]

Inductively, we have

\[N L^2(M_k)_N = (H \otimes_N H^*)^k\]

as \(N\)-\(N\) bimodules,

\[N L^2(M_k)_M = (H \otimes_N H^*)^k \otimes_N H\]

as \(N\)-\(M\) bimodules.

**Proposition 2.3** Let \(N = M_{-1} \subset M = M_0 \subset M_1 \subset M_2 \subset \cdots\) be the Jones tower. Then the derived tower is given by the spaces of \(N\)-\(N\) and \(N\)-\(M\) bimodule intertwiners. More precisely, for \(k \geq 0\),

\[N' \cap M_{2k+1} = \text{End}(N L^2(M_k)_N) = \text{End}((H \otimes H^*)^k),\]

\[N' \cap M_{2k} = \text{End}(N L^2(M_k)_M) = \text{End}((H \otimes H^*)^k \otimes H).\]

**Proof.** Since \(N \subset M_k \subset M_{2k+1}\) and \(M \subset M_k \subset M_{2k}\) are basic constructions, we have \(M_{2k+1} = \text{End}(L^2(M_k)_N)\) and \(M_{2k} = \text{End}(L^2(M_k)_M)\). With help of above observation, we have the results. \(\Box\)

Notice that \(N' \cap M_k\) are finite dimensional \(C^*\)-algebras when \(d(H) < \infty\). So the space of intertwiners is isomorphic to a multimatrix algebra as long as Jones index is finite. In this case, each direct summand matrix algebra appearing in \(N' \cap M_k\) is isomorphic to an irreducible \(N\)-\(N\) or \(N\)-\(M\) bimodule.

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The principal graph of \( N \subset M \) is defined by the Bratteli diagram of the collection of equivalent classes of irreducible \( N-N \) bimodules \( \{X\} \), \( N-M \) bimodules \( \{Y\} \) appearing as components in \( \text{End}((H \otimes H^*)^k) \) and \( \text{End}((H \otimes H^*)^k \otimes H) \), respectively. More explicitly, the principal graph of \( N \subset M \) is the bipartite graph with even vertices indexed by \( N-N \) irreducible bimodules \( \{X\} \) and odd vertices indexed by \( N-M \) irreducible bimodules \( \{Y\} \), and \( \text{dimHom}(X \otimes H, Y) \) edges between a vertex \( X \) and a vertex \( Y \).

When \( H \) is irreducible, the principal graph is connected. If the graph is finite, then \( N \subset M \) is said to have finite depth. Otherwise, it is said to have infinite depth. In other words, \( N \subset M \) has finite depth when there are only finitely many equivalent classes of irreducible \( N-N \), \( N-M \) bimodules appearing in the decompositions of the derived tower.

The depth of \( N \subset M \) is defined by the longest distance from the vertex indexed by \( N' \cap N = \text{End}(N L^2(N)_N) \) to every vertices of the principal graph.

**Remark 2.4** We can also proceed the same methods to obtain dual principal graph, starting with the associated \( M-N \) bimodule \( H^* = M L^2(M)_N \). Then

\[
\begin{align*}
M' \cap M_1 &= \text{End}(H^*) \text{ as } M-N \text{ bimodules,} \\
M' \cap M_2 &= \text{End}(H^* \otimes H) \text{ as } M-M \text{ bimodules,} \\
M' \cap M_3 &= \text{End}(H^* \otimes H \otimes H^*) \text{ as } M-N \text{ bimodules,}
\end{align*}
\]

and so on.

### 3 Applications

#### 3.1 Depth two subfactors

In this section, we describe the well-known structure of depth two subfactors \( N \subset M \) in bimodule interpretation.

**Definition 3.1** An irreducible subfactor \( N \subset M \) has depth two if \( N' \cap M_2 = \text{End}(H \otimes H^* \otimes H) \) is a factor.

It is equivalent to \( \text{dim}(N' \cap M_1) = \text{dimEnd}(H \otimes H^*) = [M : N] \) ([14]).
Lemma 3.2 ([5, 12]) Let $[M : N] < \infty$ and $N' \cap M = C I$. If $H \otimes H^* \cong \sum \oplus m_i \cdot X_i$ where $X_i$'s are irreducible $N$-$N$ submodules with $m_i = \dim \text{Hom}(X_i, H \otimes H^*)$, then the followings are equivalent:

1. $N \subset M$ have depth 2,

2. $d(X_i) = m_i$.

Proof. Notice that $H$ is an irreducible $N$-$M$ bimodule due to $N' \cap M = C I$. 

(1 ⇒ 2) Let $N \subset M$ have depth two. Then $\text{End}(H \otimes H^* \otimes H)$ is a factor. Since $H \otimes H^* \otimes H$ contains $H$ as an irreducible component, $H \otimes H^* \otimes H = [M : N]H$ with help of the multiplicative property of dimension function $d$.

On the other hand, $[M : N]H = (H \otimes H^*) \otimes H = \sum \oplus m_i(X_i \otimes H)$, and so we must have $X_i \otimes H = kH$ for some $k$. But Frobenius reciprocity implies that

$m_i = \dim \text{Hom}(X_i, H \otimes H^*) = \dim \text{Hom}(X_i \otimes H, H) = \dim \text{Hom}(kH, H) = k$.

Hence we have $X_i \otimes H = m_i H$. With this fact, the multiplicative property of the function $d$ again gives $d(X_i) = m_i$.

(2 ⇒ 1) Let $d(X_i) = m_i$. Since $[M : N] = d(H \otimes H^*) = \sum m_i d(X_i) = \sum m_i^2$,

$\dim \text{End}(H \otimes H^*) = \dim \text{Hom}(H \otimes H^* \otimes H, H)$

$= \sum m_i \dim \text{Hom}(X_i \otimes H, H)$

$= \sum m_i \dim \text{Hom}(X_i, H \otimes H^*)$

$= \sum m_i^2 = [M : N]$.

This completes the proof. □

Theorem 3.3 Let $N \subset M$ have depth two. Then $A = \text{End}(H \otimes H^*)$ is a $[M : N]$ dimensional Hopf algebra acting outerly on $M$ such that

$M_i = M \times A \quad \text{or} \quad M = N \times A^\circ$,

where $A^\circ = \text{End}(H^* \otimes H)$ denotes the dual Hopf algebra of $A$.

Proof. See [12, 14]. □

Remark 3.4 There is always a canonical pairing between $\text{End}(H \otimes H^*)$ and $\text{End}(H^* \otimes H)$, thanks to Frobenius reciprocity. In the case of depth two, this pairing provides Hopf algebra structures to $\text{End}(H \otimes H^*)$, and its dual $\text{End}(H^* \otimes H)$ as well.
4 Galois groups of subfactors

Let $N \subset M$ be factors of finite index with $N' \cap M = CI$. The Galois group for the pair is defined by

$$Gal(M, N) = \{ \alpha \in Aut(M) | \alpha|_N = id|_N \}.$$ 

When $M_1$ denotes the basic construction for $N \subset M$, the set of unitaries in $M_1$ normalizing $M$ is denoted by $N_M(M_1) = \{ u \in U(M_1) | uMu^* = M \}$. Recall (for example, see [2]) that

1. $Gal(M, N)$ is a group of order less than or equal to $[M : N]$,

2. $Gal(M, N)$ is isomorphic to $N_M(M_1)/U(M)$.

In this section we focus on computation of Galois groups based on the results in Chapter 3. For this we first discuss one dimensional $M-M$ bimodules, and then relate the Galois group to one dimensional $M-M$ bimodules appearing at some level of derived tower.

4.1 Automorphisms in bimodules

Recall that the standard Hilbert bimodule $H = M L^2(M)_M$ has dimension 1, equipped with the action $m_1 \cdot \xi \cdot m_2 = m_1 J_M m_2^* J_M \xi$ for $m_1, m_2 \in M$ and $\xi \in L^2(M)$. This provides a multiplicative unit for the relative tensor product.

Let $\alpha$ be an automorphism of $M$. Then $\alpha$ gives rise to another $M-M$ bimodule $M(\alpha H)_M$ (resp. $M(H\alpha)_M$) of dimension 1, which is the same Hilbert space $L^2(M)$ equipped with the action

$$m_1 \cdot \xi \cdot m_2 = \alpha(m_1) J_M m_2^* J_M \xi,$$

(resp., $m_1 \cdot \xi \cdot m_2 = m_1 J_M \alpha(m_2)^* J_M \xi$),

where $m_1, m_2 \in M$.

It is clear that $\alpha H \cong \beta H$ if and only if there is a unitary $u \in M$ such that $\alpha = (Ad u) \beta$ on $M$, for $\alpha, \beta \in Aut(M)$.

Lemma 4.1 ([5, 8]) Let $M$ be a type $II_1$ factor and $\alpha, \beta \in Aut(M)$. Then we have the following ;
1. \( ^{\alpha}H \cong H^{\alpha^{-1}} \),

2. \((^{\alpha}H)^* \cong H^\alpha\),

3. \( ^{\alpha}H \otimes_M \beta H \cong \beta \alpha H, \ H^\alpha \otimes_M H^\beta \cong H^{\alpha\beta} \),

4. \( ^{\alpha}H \cong H \) if and only if \( \alpha \in \text{Inn}(M) \).

Note that Lemma 4.1 implies that \( ^{\alpha}H \) is self-conjugate if and only if \( \alpha^2 \in \text{Inn}(M) \). Finally we have the following ([10, 9]).

**Proposition 4.2** A bimodule \( _MX_M \) has \( d(X) = 1 \) if and only if there exists an outer automorphism \( \alpha \in \text{Aut}(M) \) so that \( X \cong ^{\alpha}H \).

**Proof.** For \( x \in X \), let \( m \cdot x = \pi^X_l(m)x \), \( x \cdot m = \pi^X_r(m)x \) be the left and right representations of \( M \) on \( X \), while \( M \) has a standard form on \( L^2(M) \), with left and right multiplications \( m \cdot \xi = \pi_l(m)\xi \), \( \xi \cdot m = \pi_r(m)\xi \) respectively, for \( \xi \in L^2(M) \).

It is enough to show the necessity. Assume that \( _MX_M \) has \( d(_MX_M) = 1 \). Then the coupling constant \( \text{dim}_M X = \text{dim}_M X = 1 \), and hence \( M \) has a standard form on \( X \) as well. Therefore there is a unitary \( u : X \rightarrow L^2(M) \) such that \( \pi_r(m) = u\pi^X_l(m)u^* \), for \( m \in M \).

Taking \( \alpha = \pi_l^{-1} \circ \text{Ad}u \circ \pi^X_l \), we see that \( \alpha \in \text{Aut}(M) \) and \( \pi_l(\alpha(m)) = u\pi^X_l(m)u^* \). Moreover, if \( x \in X \),

\[
u(m \cdot x \cdot n) = u\pi^X_l(m)\pi^X_r(n)x = \pi_l(\alpha(m))uu^*\pi_r(n)ux = \alpha(m)J_Mn^*J_M \cdot ux.\]

Therefore, via \( u \), \( X \cong ^{\alpha}H \) as \( M-M \) bimodules. It now follows from Lemma 4.1 that \( \alpha \) is not inner. \( \square \)

### 4.2 Galois groups and grouplike elements

Let \( N \subset M \) have depth two. Then \( \text{End}(H^* \otimes H) \) is a \([M : N]\) dimensional Hopf algebra. In this section, we study irreducible \( M-M \) bimodules of dimension 1 appearing in the Hopf algebra \( \text{End}(H^* \otimes H) \). We call them the grouplike elements.

Notice that if two irreducible A-B bimodules \( X \) and \( Y \) satisfy \( \text{Hom}(X, Y) \neq 0 \), then \( X \cong Y \) as A-B bimodules.
Proposition 4.3 ([10]) Let $N' \cap M = CI$. Then the set of the grouplike elements in $\text{End}(H^* \otimes H)$ forms a group under relative tensor product, with identity $M \cdot H_M$. Moreover this group is isomorphic to the Galois group $G(M, N)$.

Proof. If follows from Proposition 4.2 that there exists outer automorphism $\alpha \in \text{Aut}(M)$ corresponding to a 1 dimensional $M \cdot M$ submodule appearing in $\text{End}(H^* \otimes H)$. Then the group structure on the set of grouplike elements comes from Lemma 4.1 and Proposition 4.2.

Note that Frobenius reciprocity implies

$$0 \neq \text{Hom}(M \cdot H_M, M \cdot H^* \otimes N \cdot H_M) \cong \text{Hom}(N \cdot H \otimes M \cdot H_M, N \cdot H_M)$$

$$\cong \text{Hom}(N \cdot H \otimes M \cdot H^{-1} \cdot M, N \cdot H_M) \cong \text{Hom}(N \cdot H_M, N \cdot H_M),$$

when $N \cdot H_M$ with restricted left action of $\alpha$ to $N$. But $\text{End}(N \cdot H_M) = \alpha(N)' \cap M \cong C$, we see that $N \cdot H_M$ is an irreducible $N \cdot M$ bimodule. Thus, as $N \cdot M$ bimodules, $N \cdot H_M \cong N \cdot H_M$, via a unitary $v$. Since $v$ commute with the left action, $v \alpha(n) = n v$ for $n \in N$, and hence $(Ad v) \alpha|_N = id_N$. This completes the proof. \( \square \)

Consequently, the Galois group $G(M, N)$ is determined by 1 dimensional $M \cdot M$ submodules appearing in $\text{End}(H^* \otimes H)$. Note that the condition $N' \cap M = CI$ in Proposition 4.3 is essential (for the counter example, see [10]).

We are now going to calculate the Galois groups in various cases ([3, 4, 6, 10]).

Example 4.4 Let a finite group $G$ act outerly on a type $II_1$ factor $N$ via $\alpha$. Let $M$ be the crossed product $M = N \times_\alpha G$. Then the associated $N \cdot M$ bimodule $H = N \cdot L^2(M) \cdot M$ is irreducible, due to the outerness of the action. Moreover, the relative commutants are easily expressed as

$$N \cdot H \otimes M \cdot H_N^* = N \cdot L^2(M) \cdot N = \bigoplus_{g \in G} \alpha_g L^2(N),$$

with mutually non-equivalent irreducible $N \cdot N$ bimodules $\alpha_g L^2(N)$ of dimension 1.

On the other hand,

$$M \cdot H^* \otimes N \cdot H_M = M \cdot L^2(M_1) \cdot M = \bigoplus_{\sigma \in G^*} \otimes X^\sigma_j,$$
where $G^*$ denotes the set of representations of $G$. Here $\mathcal{M}-\mathcal{M}$ bimodule $X_j^\sigma = \sum_i \oplus L^2(\mathcal{M})\xi_{i,j}^\sigma$ is an irreducible $\mathcal{M}-\mathcal{M}$ bimodule with linear basis defined by $\xi_{i,j}^\sigma = \sum_{g \in G} \sigma_{i,j}(g^{-1})u_{g^{-1}} \otimes u_g \in H^* \otimes H$. Therefore $X_j^\sigma$ is one dimensional bimodule if and only if $\sigma$ is one dimensional representation of $G$, so the Galois group $\mathcal{G}^{\mathcal{M},N}$ is group-isomorphic to the character group $\hat{G} \cong G/[G,G]$.

**Example 4.5** Let $N = \mathcal{M}^G \subset \mathcal{M}$. Then $H =_N L^2(\mathcal{M})_M$ is also irreducible bimodule and

\[ N H \otimes H^*_N =_M L^2(\mathcal{M})_M = \sum \oplus_{\sigma \in G^*} X^\sigma, \]

where $G^*$ denotes the set of representations of $G$. Also

\[ M H^* \otimes H_M =_M L^2(\mathcal{M}_1)_M = M L^2(\mathcal{M} \times \mathcal{G})_M = \sum \oplus_{\alpha \in \mathcal{G}} L^2(\mathcal{M}). \]

We see that each element $g \in G$ is associated with a 1 dimensional bimodule $\alpha_g L^2(\mathcal{M})$, therefore Galois group $\mathcal{G}^{\mathcal{M},N}$ is group-isomorphic to the group $G$.

**Example 4.6** Let finite groups $K \subset G$ act outerly on a type $II_1$ factor $\mathcal{P}$. We consider $N = \mathcal{P} \times K \subset \mathcal{P} \times G = \mathcal{M}$. Let $s = KsK$ be the K-K double coset representatives in $K \backslash G / K$. If we let $K_s = K \cap sKs^{-1}$, then

\[ N H \otimes H^*_N =_{P \times K} L^2(\mathcal{P} \times \mathcal{G})_{P \times K} = \sum_s \sum_{i \in K_s} \oplus Y_s^i, \]

where $i$ denotes the identity representation of $K_s$. On the other hand, if we denote $\overline{G} = \{ \sigma \in G^* \mid \sigma|_K = i_K \}$,

\[ M H^* \otimes H_M =_{P \times G} L^2(\mathcal{M}_1)_{P \times G} = \sum_{\sigma \in \overline{G}} \oplus X_\sigma. \]

Since $X_\sigma$ is one dimensional if and only if $\dim(\sigma) = 1$, the Galois group is isomorphic to the characters of $G$ vanishing on $K$.

**Example 4.7** Under the same condition as Example 4.6, we now consider $N = \mathcal{P}^G \subset \mathcal{P}^K = \mathcal{M}$. Then the role of $N-N$, $\mathcal{M}-\mathcal{M}$ bimodules are switched in this case, i.e.,

\[ N H \otimes H^*_N =_{P^G} L^2(\mathcal{P}^K)_{P^G} = \sum_{\sigma \in \overline{G}} \oplus X_\sigma, \]
and

\[ M H^* \otimes H_M =_{P_K} L^2(M_1)_{P_K} = \sum_s \sum_{i \in K_s} \oplus Y^i_s, \]

where \(i\) denotes the identity representation of \(K_s\). Then irreducible bimodule \(Y^i_s\) has dimension one if and only if \(K = K_s\), or \(s \in \mathcal{N}_G(K)\); in other words, those \(K_s\) satisfying \(K = K_s\) are paramatized by \(\mathcal{N}_G(K)/K\). Thus the Galois group is isomorphic to \(\mathcal{N}_G(K)/K\).

**Example 4.8** Let \(H\) and \(K\) be subgroups in a finite group \(G\) such that \(H \cap K = \{e\}\) and \(HK\) be a group. Let \(\alpha\) and \(\beta\) be the outer actions of \(H\) and \(K\) respectively on a type \(II_1\) factor \(P\). We consider the subfactors \(N = P^H \subset P \times_{\alpha} K = M\). We know that \(N \subset M\) is irreducible and have depth two in this case ([1, 3, 6]).

First, let \(A =_N L^2(P)_P\), \(B =_P L^2(M)_M\), and \(X = A \otimes_P B =_N L^2(M)_M\) be the associated irreducible \(N\)-\(M\) bimodule. Note that

\[ A^* \otimes A =_P L^2(P) \otimes_N L^2(P)_P =_P L^2(P \times H)_P = \sum \oplus_{h \in H} [\beta_h], \]

and

\[ B \otimes B^* =_P L^2(M) \otimes_M L^2(M)_P =_P L^2(P \times K)_P = \sum \oplus_{s \in K} [\alpha_s], \]

where \([\theta] =_P \theta L^2(P)_P\) with the action \(m \cdot [\xi] \cdot n = \theta(m) J_P n^* J_P \xi\) for \(n, m \in P\), and an outer automorphism \(\theta\) of \(P\).

For the Galois group, we need to compute the submodules of dimension one appearing in \(X^* \otimes X\). Since

\[ X^* \otimes X =_M (B^* \otimes_P A^*) \otimes_N (A \otimes B)_M \]
\[ =_M B^* \otimes_P (\sum \oplus_{h \in H} [\beta_h]) \otimes_P B_M \]
\[ = \sum \oplus_{h \in H} M(B^* \otimes_P [\beta_h] \otimes_P B)_M, \]

we have

\[ \text{End}(X^* \otimes X) = \text{Hom}(X^* \otimes X, X^* \otimes X) \]
\[ = \sum \oplus_{g, h \in H} \text{Hom}(B^* \otimes_P [\beta_g] \otimes_P B, B^* \otimes_P [\beta_h] \otimes_P B). \]
On the other hand, for $g, h \in H$,

\[ \text{Hom}(B^* \otimes P [\beta_g] \otimes P B, B^* \otimes P [\beta_h] \otimes P B) = \text{Hom}([\beta_g], B \otimes P B^* \otimes P [\beta_h] \otimes P B \otimes P B^*) \]

\[ = \sum \oplus_{s,t \in K} \text{Hom}([\beta_g], [\alpha_s] \otimes [\beta_h] \otimes [\alpha_t]). \]

But \( \text{Hom}([\beta_g], [\alpha_s] \otimes [\beta_h] \otimes [\alpha_t]) = 0 \) for some \( s, t \in K \) if and only if \( g \neq sht \) in \( \text{Out}(P) \). i.e., \( B^* \otimes P [\beta_g] \otimes P B \) and \( B^* \otimes P [\beta_h] \otimes P B \) are disjoint if and only if \( KgK \neq KhK \). In other words, the double cosets \( KhK \) of \( K \backslash G / K \) parameterize nonequivalent bimodules \( B^* \otimes P [\beta_h] \otimes P B \). Therefore we have

\[ \text{End}(X^* \otimes X) = \sum_{h=KhK} \oplus \text{End}(B^* \otimes P [\beta_h] \otimes P B) \]

\[ = \sum_{h=KhK} \sum_{s,t \in K} \oplus \text{Hom}([\beta_h], [\alpha_s] \otimes [\beta_h] \otimes [\alpha_t]), \]

where \( h = KhK \in K \backslash G / K \) is the representative of \( K \)-\( K \) double cosets in \( G \).

But \( \text{Hom}([\beta_h], [\alpha_s] \otimes [\beta_h] \otimes [\alpha_t]) \neq 0 \) if and only if \( ths = h \), or \( t = hs^{-1}h^{-1} \in hKh^{-1} \cap K \). Therefore

\[ \text{End}(X^* \otimes X) = \sum_{h=KhK} \sum_{s \in K \cap hKh^{-1}} \oplus \text{Hom}([\beta_h], [\alpha_s] \otimes [\beta_h] \otimes [\alpha_{hs^{-1}h^{-1}}]) \]

\[ \cong \sum_{h=KhK} \oplus C_\omega[K \cap hKh^{-1}], \]

the direct sum of twisted group algebras. Therefore the Galois group is determined by one dimensional irreducible representations of \( K_h = K \cap hKh^{-1} \), for all \( h \in K \backslash G / K \).

**Lemma 4.9** \( \chi \in \hat{K}_h \) has dimension 1 if and only if \( h \in \mathcal{N}_H(K) \), where \( \mathcal{N}_H(K) = \{ h \in H \mid hKh^{-1} = K \} \) denotes the normalizer subgroup of \( K \) in \( H \).

**Proof.** For \( \chi \in \hat{K}_h \), note that

\[ \frac{\dim \chi \cdot |KhK|}{|K|^2} = (\dim \chi)^2 \cdot \frac{|K|}{|K_h|}. \]

Since \( |KhK| = |K|^2 / |K_h| \), \( \dim \chi = \frac{|K_h|}{|K|} \). Thus \( \dim \chi = 1 \) if and only if \( |K| = |K_h| \), if and only if \( h \in H \) satisfies \( hKh^{-1} = K \), or \( h \in \mathcal{N}_H(K) \). \( \square \)
With Lemma 3.9, we finally have

$$G(M, N) = \sum_{h \in N_H(K)} \oplus C_{\omega}(K).$$

Now the mapping of $h \in N_H(K)$ into a map $(\chi \rightarrow \chi^h)$ determines an action of $N_H(K)$ on $\hat{K}$, where $\chi^h(k) = \chi(h^{-1}kh)$ for $\chi \in \hat{K}$. Hence the Galois group $G(M, N)$ is isomorphic to the semidirect product group $\hat{K} \times N_H(K)$.

References


Department of Applied Mathematics, Korea Maritime University, Pusan 606-791, KOREA