

The Existence and Uniqueness of Solution to the Volterra Integrodifferential Equation of Parabolic Type

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1. Introduction

We consider nonlinear heat flow in a homogeneous bar of unit length of material with memory with the temperature $u=u(t, x)$ maintained at zero at $x=0$ and $x=1$ by the similar method as [2]. We shall assume that the history of u is prescribed for $t \geq 0$ and $0 \leq x \leq 1$. The equation satisfied by u in such a material is derived from assumptions that the internal energy ϵ and the heat flux q are functionals of u and of the gradient of u , respectively. According to the theory developed by Nunziato [13] for heat flow in materials of fading memory type the functionals ϵ and q are taken respectively as

$$\epsilon(t, x) = \alpha u(t, x) + \int_0^t \beta(t-s)u(s, x)ds, \quad (1.1)$$

$$q(t, x) = -ku_x(t, x) - \int_0^t a(t-s)\sigma(u_x(s, x))ds \quad (1.2)$$

where $\alpha > 0, k > 0$ are given constants and $\beta, a : [0, \infty] \rightarrow \mathbb{R}$ are given sufficiently smooth functions (called the internal energy and heat flux relaxation functions, respectively). In the physical literature β and a are usually assumed to be decaying exponentials with positive coefficients. The real function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ in (1.2) will be assumed to $\sigma \in C^1(\mathbb{R}), \sigma(0) = 0$. It should be noted that the $\sigma(r) \equiv r$ gives rise to the linear model derived in Nunziato [13], and that (1.2) is one reason-

able generalization of the heat flux for nonlinear heat flow in one space dimension.

If $f = f(t, x) \in C^1([0, \infty) ; L^1(0, 1))$ represents the external heat supply added to the rod for $t \geq 0$ and $0 \leq x \leq 1$, and if $u(0, x) = u_0(x), 0 \leq x \leq 1$ is the given initial temperature distribution, the law of balance of heat ($\epsilon_t = -\text{div}q + f$) shows that in one space dimension the temperature u satisfies the initial boundary value problem

$$\begin{aligned} \frac{\partial}{\partial t} \{ \alpha u(t, x) + \int_0^t \beta(t-s)u(s, x)ds \} \\ - ku_{xx}(t, x) - \int_0^t a(t-s)(\sigma(u_x(s, x)))_x ds \\ = f(t, x), \end{aligned} \quad (1.3)$$

$u(0, x) = u_0(x) (0 \leq x \leq 1), u(t, 0) = u(t, 1) \equiv 0 (t \geq 0)$, where subscripts denote differentiation with respect to x .

In the case $\beta(t) \equiv 0, \alpha = \kappa = 1$, in Section 3 we consider the initial boundary value problem to the following equation :

$$\begin{aligned} u_t(t, x) - u_{xx}(t, x) - \int_0^t a(t-s)(\sigma(u_x(s, x)))_x ds \\ = f(t, x), 0 \leq x \leq 1, t \geq 0, \end{aligned} \quad (1.4)$$

$$u(t, 0) = u(t, 1) = 0, t \geq 0, \quad (1.5)$$

$$u(0, x) = u_0(x), 0 \leq x \leq 1. \quad (1.6)$$

2. Preliminaries

Here, all functions may be real value. We denote by $L^2(0, 1)$ consist of functions f on some

interval $[0, 1]$ such that $|f(t)|^2$ is Lebesgue integrable on this interval. The space $L^2(0, 1)$ is Hilbert space with the inner product

$$(f, g) = \int_0^1 f(x)g(x)dx,$$

and norm is given by

$$\|f\| = (\int_0^1 f(x)^2 dx)^{1/2}$$

$H^1(0, 1)$ denotes the set of all functions whose u is absolutely continuous on $[0, 1]$ and whose derivatives up to degree 1 belong to $L^2(0, 1)$. $H^1(0, 1)$ is a Hilbert space with inner product and norm by

$$\begin{aligned} ((u, v))_1 &= (u', v') + (u, v), \\ \|u\|_1 &= (\|u'\|^2 + \|u\|^2)^{1/2}, \end{aligned}$$

respectively, for all $u, v \in H^1(0, 1)$.

Denotes

$$\begin{aligned} H_0^1(0, 1) &= \{u \in H^1(0, 1) \mid u(0) = u(1) = 0\}, \\ H_0^1(0, 1) &\subset H^1(0, 1). \end{aligned}$$

Hence $H_0^1(0, 1)$ is a Hilbert space with inner product and norm by

$$\begin{aligned} ((u, v)) &= \int_0^1 u'(x)v'(x)dx, \\ \|u\| &= (\int_0^1 u'(x)^2 dx)^{1/2}. \end{aligned}$$

respectively, for all $u, v \in H_0^1(0, 1)$. Denotes $H^2(0, 1) = \{u \in H^1(0, 1) \mid u' \text{ ; absolutely continuous on } [0, 1], u'' \in L^2(0, 1)\}$. Hence $H^2(0, 1)$ is a Hilbert space with inner product and norm

$$\begin{aligned} ((u, v))_2 &= (u'', v'') + (u', v') + (u, v), \\ \|u\|_2 &= (\|u''\|^2 + \|u'\|^2 + \|u\|^2)^{1/2}, \end{aligned}$$

respectively, for all $u, v \in H^2(0, 1)$. $H^{-1}(0, 1) = H_0^1(0, 1)^*$ denotes the set of all functions whose derivatives up to degree 1 in distribution sense belong to $L^2(0, 1)$. i.e.,

$$H^{-1}(0, 1) = \left\{ \frac{df}{dx} \mid f \in L^2(0, 1) \right\}$$

where $H_0^1(0, 1)^*$ is dual space of $H_0^1(0, 1)$. By the definition, if $u \in H^{-1}(0, 1)$ then there exist $f \in$

$L^2(0, 1)$ such that $u = f'$ but f don't exist uniquely. As a matter of fact, let $u = f'$ and c is a constant then $f + c \in L^2(0, 1)$,

$$u = (f + c)'.$$

Conversely, suppose that $f, g \in L^2(0, 1)$ and $f' = g' = u$. then $(f - g)' = 0$. Hence f is constant function. If we select f such that $\int_0^1 f(x)dx = 0$, then f is uniquely determined by u . Let

$$\begin{aligned} u, v \in H^{-1}(0, 1), u = f', v = g', \int_0^1 f(x)dx \\ = \int_0^1 g(x)dx = 0 \end{aligned}$$

and

$$((u, v)) \cdot = (f, g), \|u\| \cdot = \|f\|,$$

then $H^{-1}(0, 1)$ becomes a Hilbert space with inner product $((\cdot, \cdot)) \cdot$ and norm $\|\cdot\| \cdot$. Therefore,

$$H_0^1(0, 1) \subset L^2(0, 1) \subset H^{-1}(0, 1).$$

3. Assumptions and main theorem

By the definition,

$$H^2(0, 1) \cap H_0^1(0, 1) = \{u \in H^2(0, 1) \mid u(0) = u(1) = 0\}.$$

Let A be the operator defined by

$$D(A) = \text{domain of } A = \{u \mid u \in H^2(0, 1) \cap H_0^1(0, 1)\}, \quad (3.1)$$

$$Au = -\Delta u (\Delta = \text{Laplacian}), u \in D(A). \quad (3.2)$$

Then A is positive definite self-adjoint operator. suppose that $\sigma(r)$ is defined continuously differentiable on R , the derivative $\sigma'(r)$ is bounded. That is, there exists $M > 0$ such that

$$|\sigma'(r)| \leq M \quad (3.3)$$

for $-\infty < r < \infty$. Obviously $\sigma(r)$ satisfies uniformly Lipschitz condition. In other words,

$$|\sigma(r) - \sigma(s)| \leq M|r - s| \quad (3.4)$$

for $r, s \in R$. For $u \in D(A)$ and by (3.4), the following inequality holds.

$$|\sigma(u'(x)) - \sigma(u'(y))| \leq M |u'(x) - u'(y)|.$$

Hence $\sigma(u'(x))$ is absolutely continuous. For $u \in D(A)$, the following equality holds

$$\sigma(u'(x))' = \sigma'(u'(x))u''(x). \quad (3.5)$$

Defining $g(u(x)) = (\sigma(u'(x)))'$ then $g(u(x)) = \sigma'(u'(x))u''(x)$. By virtue of (3.3), for each $u \in D(A)$, it follows that

$$g(u) \in L^2(0, 1), |g(u)| \leq M |u''|.$$

We need the following assumptions :

(i) $a(t)$ is Hölder continuous on $[0, \infty)$ with exponent ρ , that is, there exists a constant $c > 0$ and $0 < \rho \leq 1$ such that

$$|a(t) - a(s)| \leq c |t - s|^\rho$$

for all $t, s \in [0, \infty)$.

(ii) $u_0 \in D(A)$.

(iii) $f(t) \in C^1([0, \infty) ; L^2(0, 1))$.

We write the mixed problem (1.2) - (1.4) as a formulation in $L^2(0, 1)$,

$$\frac{du}{dt}(t) + Au(t) \int_0^t a(t-s)g(u(s))ds = f(t), \quad (3.6)$$

$$u(0) = u_0. \quad (3.7)$$

Theorem 3.1. Assume that (i) - (iii) hold. Let $u(t), f(t)$ be a $L^2(0, 1)$ -valued functions of t , respectively and

$$u(t) \in D(A), \frac{du}{dt}(t) = \lim_{h \rightarrow 0} \frac{u(t+h) - u(t)}{h}$$

be exist in the $L^2(0, 1)$ -topology. Then there exist a unique solution $u(t)$ of (3.6), (3.7).

We can solve the problem (3.6), (3.7) by the following method. Setting $q(v, u) = \sigma'(v)u''$ for all $v \in H_0^1(0, 1)$ and for all $u \in D(A)$. Obviously, $q(u, u) = g(u)$.

(a) Let $v(t)$ be a $H_0^1(0, 1)$ -valued continuous function such that $v(0) = u_0$. We solve that the following initial value problem :

$$\frac{du}{dt}(t) + Au(t) \int_0^t a(t-s)q(v(s), u(s))ds = f(t), \quad (3.8)$$

$$u(0) = u_0. \quad (3.9)$$

(b) Since the solution of Eqs. (3.8), (3.9) is exist and unique, the mapping θ defined from $H_0^1(0, 1)$ to $D(A)$ i.e., $\theta v = u$. Therefore there exists a fixed point of θ , that is, the fixed point is a solution of Eqs.(3.6), (3.7).

Proof of the existence [7, 8], (a) Obviously, $q(v, u)$ satisfies uniformly Lipschitz condition that the following inequality : there exists $M > 0$ such that

$$|q(v, u_1) - q(v, u_2)| \leq M \|u_1 - u_2\|$$

for all $v \in H_0^1(0, 1)$ and $u_1, u_2 \in D(A)$. Since A is a positive definite self-adjoint operator, we set

$$A = \int_0^\infty \lambda dE(\lambda); \text{ (real spectral resolution of operator } A,$$

where E is a real spectral measure.

We are defined $\exp(-tA) = \int_0^\infty \exp(-\lambda t)E(\lambda)$ for all $t \geq 0$. It is known that, $-A$ generates an analytic semigroup in $L^p(0, 1)$ ($1 \leq p < \infty$). For each $v \in D(A)$ we define

$$\begin{aligned} (K_v u)(t) &= \exp(-tA)u_0 \\ &+ \int_0^t \exp(-(t-s)A)f(s) \\ &- \int_0^s a(s-r)q(v(r), u(r))dr ds. \end{aligned}$$

We show that for t_0 sufficiently small, K_v is a contraction. By (3.3), there exists a constant $c(0 < c < 1)$ that satisfy a following inequality :

$$\|(K_v u_1)(t) - (K_v u_2)(t)\| \leq c \|u_1(t) - u_2(t)\|$$

for all $0 \leq t \leq t_0$. It follows that K_v is a contraction mapping of $D(A)$ into $H^2(0, 1)$. Hence there is a unique fixed point u of K_v in $H^2(0, 1)$ and $u(t)$ is a local solution of (3.8), (3.9). Then by

previous arguments there exists a unique solution $u(t)$ of (3.8), (3.9) on $[0, \infty)$.

(b) Let t_1, L, η are positive number and $\eta < 1$. Define

$$S = \{v \in H_0^1(0, 1) \mid \|v(t) - v(s)\| \leq L|t - s|^\eta, v(0) = u_0 \text{ in } [0, t_1]\}.$$

For t_1 , sufficiently small, then $\theta : S \rightarrow S$ is continuously compact mapping for some L, η . Since S is closed, bounded, and convex, it follows from Schauder's fixed point theorem there is a unique fixed point that satisfies (3.6), (3.7) on $[0, t_1]$. By the continuity, there exists a solution of (3.6), (3.7) on $[0, \infty)$.

Proof of the Uniqueness. I the previous proof we have only the existence of solution of Eqs. (1.2) - (1.4) in $L^2(0, 1)$. It remains to prove the uniqueness of (1.2) - (1.4). For this purpose we need some elements of functional analysis. Putting

$$\bar{A}u = -u'' \text{ for all } u \in H_0^1(0, 1).$$

Since $-u'' = (-u)'$, $-u' \in L^2(0, 1)$ hold then

$$\bar{A}u \in H_0^1(0, 1), \int_0^1 -u'(x)dx = -u(1) + u(0) = 0.$$

Hence, if for $v \in H_0^1(0, 1)$, the function g satisfies

$$g' = v, \int_0^1 g(x)dx = 0,$$

then by integration by parts and $u(0) = u(1) = 0$.

$$((\bar{A}u, v)) = (-u', g) = (u, g') = (u, v),$$

$$((u, \bar{A}v)) = ((\bar{A}v, u)) = (v, u) = (u, v) = ((\bar{A}u, v)).$$

Therefore, the operator A is a symmetric and a positive definite self - adjoint on $H^{-1}(0, 1)$. Defined the linear operator $\bar{A} : H_0^1(0, 1) \rightarrow H^{-1}(0, 1)$ by

$$D(A) = \{u \in H_0^1(0, 1) \mid \bar{A}u \in L^2(0, 1)\}, \quad (3.10)$$

$$\bar{A}u = Au \text{ for all } u \in D(A). \quad (3.11)$$

Since we may replace $\sigma(r)$ by $\sigma(r) + \text{constants}$ without altering equation (1.4), we may assume that $\sigma(0) = 0$, by (3.4) for all $u \in H_0^1(0, 1)$ then

$$|\sigma(u'(x))| = |\sigma(u'(x)) - \sigma(0)| \leq M|u'(x)|.$$

Hence

$$\sigma(u') \in L^2(0, 1), (\sigma(u')) \in H^{-1}(0, 1).$$

And if we put $\bar{g}(u) = (\sigma(u'))'$ then \bar{g} is a mapping from $H_0^1(0, 1)$ to $H^{-1}(0, 1)$. In particular, if $u \in D(A)$, then $\bar{g}(u) = g(u)$. By the above argument, we consider the mixed problem (1.4) - (1.6) as a formulation in $H^{-1}(0, 1)$;

$$\frac{du}{dt}(t) + \bar{A}u(t) + \int_0^t \alpha(t-s)\bar{g}(u(s))ds = f(t), \quad (3.12)$$

$$u(0) = u_0 \quad (3.13).$$

Obviously, if the solution of initial value problem (3.6) - (3.7) exists then that of (3.12) - (3.13) exists, and the solution of initial value problem (3.12) - (3.13) is unique then that of (3.6) - (3.7) is unique.

We prove that the solution (3.12) - (3.13) is unique. In general, if $f \in L^2(0, 1)$ then $\|f'\| \leq \|f\|$. By (3.4), we have

$$\begin{aligned} \|\bar{g}(u) - \bar{g}(v)\| &= \|\sigma((u'))' - (\sigma(v'))'\| \\ &\leq |\sigma(u') - \sigma(v')| \\ &\leq M|u' - v'| = M\|u - v\|. \end{aligned}$$

Hence \bar{g} satisfies a uniform Lipschitz condition. Therefore the uniqueness of solution of (3.12), (3.13) follows from (3.8), (3.9).

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