Integrodifferential Equations of Sobolev Type

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1. Introduction

In this paper, our aim is to investigate the local existence, uniqueness and asympototic behavior of solution of functional integrodifferential equations of the more general type which involve a nonlinear delay term in a Banach space. More precisely we consider functional integrodifferential equation of the form :

$$(Bx(t))'+Ax(t)=g(t, x_t, \int_0^t k(t,s,x_s)ds),\ t\in[0, T]$$
$$x(t)=\phi(t),\ t\in[-r, 0] \quad (1.1)$$

where g, k are nonlinear continuous functions and A and B are closed linear operators with domains contained in a Banach space X and ranges in a Banach space Y. Again, consider the following equation concerned with above equation

$$y'(t)+AB^{-1}y(t)$$
$$=g(t, B^{-1}y_t, \int_0^t k(t,s,B^{-1}y_s)ds),\ t\in[0, T]$$
$$y(t)=B\phi(t),\ t\in[-r, 0] \quad (1.2)$$

where $\phi\in D(B)$. Equations of Sobolev type have been studied by several mathematicians. But we shall study more general equation as (1.1). Our proof technique is different from paper[4] tried by approximate solutions. And A. G. Karsatos and M. E. Parrott[5] have dealt pseduoparabolic problems with operator $A(t, u_t)$.

2. Preliminaries

Let X be a real Banach space with norm $\|\cdot\|_x$, denote by Y be a real Banach space with norm $\|\cdot\|$, and intervals be used at this paper are $[-r, T]$ and $[0, T]$ where $r>0$ and $T>0$ are constants. To obtain a result, we give as the following conditions.

$(C1)$ The operators $A : D(A)\subset X\to Y$ and $B : D(B)\subset X\to Y$ satisfy the following facts :

(i) A and B are closed linear operators,

(ii) $D(B)\subset D(A)$ and B is bijective,

(iii) $B^{-1} : Y\to D(B)$ is a continuous operator.

The hypotheses(i), (ii), and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : Y\to Y$. For a continuous function $x : [-r, T]\to$ X(resp. Y), x_t is that element of $C=C([-r, 0]; X)$(resp. $\bar{C}=C([-r, 0]; Y)$) defined by $x_t(\theta)=x(t+\theta)$, $-r\le\theta\le 0$. The domain D(B) of B becomes a Banach space with norm $\|x\|_{D(B)}=\|Bx\|$, $x\in D(B)$ and $C(B)=C([-r, 0]; D(B))$. The supnorms of C, $\bar{C}$ and $C(B)$ will be denoted, respectively, by $\|\cdot\|_C$, $\|\cdot\|_{\bar{C}}$, and $\|\cdot\|_{C(B)}$.

(C_{21}) $\|T(t)\|\le Me^{wt}$

(C_{22}) $\|T(t)\|\le Me^{-wt}$ where constant $M\ge 1$ and $w>0$.

(C_3) $g : J_0\times C(B)\times X\to Y$ is nonlinear continuous operator.

$$\|g(t,\psi,x)-g(t,\bar{\psi},\bar{x})\|\le L_1(t)$$
$$[\|\psi-\bar{\psi}\|_{C(B)}+\|x-\bar{x}\|_x]$$

(C_{41}) $k : J_0 \times J_0 \times C(B) \rightarrow Y$ is nonlinear continuous operator.

$$\| k(t, s, \psi) - k(t, s, \bar{\psi}) \| \leq L_2(s) e^{w(t-s)} \| \psi - \bar{\psi} \|_{C(B)}$$

(C_{42})

$$\| k(t,s,\psi) - k(t,s,\bar{\psi}) \| \leq L_2(s) e^{-w(t-s)} \| \psi - \bar{\psi} \|_{C(B)}$$

$g(t, 0, 0) = 0, k(t, s, 0) = 0,$

where $L_1, L_2 \in C(R^+, R^+)$ and denote $J_0 = [0, T]$.

By the variation of parameters formula, we obtain integral equation

$$y(t) = T(t)B\phi(0) + \int_0^t T(t-s)g(s, B^{-1}y_s, \int_0^s k(s, \tau B^{-1}y_\tau)d\tau)ds, \ t \in [0, T]$$
$$y(t) = B\phi(t), \ t \in [-r, 0] \quad (2.1)$$

(2.1) is called mild solution of (1.2). where $T(t)$ is the semigroup of bounded linear operators generated by $-AB^{-1}$

Definition 2.1.

A solution $x(t)$ of equation (1.1) is a continuous function defined on $[-r, T] \rightarrow X$ for some $T > 0$ such that $x(t) \in D(A)$ and $x'(t) \in D(B)$ for all $t \in (0, T]$, $Ax \in C([0, T] ; Y)$, $Bx' \in C((0, T] ; Y)$ and equation (1.1) holds for all $t \in [-r, T]$.

Definition 2.2.

The solution $x(t)$ of (1.1) is said to be exponentially asymptotically, if there exist positive constants N and w such that the inequality

$$\| x_t \|_{C(B)} \leq N \| \phi \|_{C(B)} e^{-wt}, \ t \geq 0$$

holds for $\| \phi \|_{C(B)}$ sufficiently small.

Lemma 2.3[6].

Let $a(t)$, $b(t)$ and $c(t)$ be real - valued nonnegative continuous functions defined on R^+, for which the inequality

$$c(t) \leq c_0 + \int_0^t a(s)c(s)ds + \int_0^t a(s)[\int_0^s b(\tau)c(\tau)d\tau]ds,$$

holds for all $t \in R^+$, where c_0 is nonnegative constant. Then

$$c(t) \leq c_0[1 + \int_0^t a(s)\exp[\int_0^s (a(\tau) + b(\tau)d\tau]ds,$$

for all $t \in R^+$.

In order to prove Theorem 3.1 in section 3, first of all, say the following fact : since AB^{-1} is a bounded linear operatdr, a function y(t) is a solution of Eq.(1.2) if and only if it is a solution of Eq.(2.1) if and only if $x(t) = B^{-1}y(t)$ is a solution of Eq.(1.1).

3. Result

Theorem 3.1.

Suppose (C_1), (C_{21}), (C_3), (C_{41}) hold. For each $\phi \in C(B)$, there is a unique continuous function $x : [-r, T] \rightarrow X$ satisfying

$$x(t) = T(t)B\phi(0) + \int_0^t T(t-s)g(s, x_s, \int_0^s k(s, \tau, x_\tau)d\tau)ds,$$
$$t \in [0, T], \ x(t) = \phi(t), \ t \in [-r, 0]$$

provided that

$$MK_1Te^{wT}[1 + K_2T][\gamma + \| \phi \|_{C(B)}]/\gamma,$$
$$MK_1T[1 + K_2T]e^{wt} < 1.$$

Proof. Choose $\gamma > 0$ such that

$$H = \{\psi \in C([-r, T] ; Y) : \psi(0) = B\phi(0), \| \psi - B\phi \|_C \leq, 0 \leq t \leq T\}.$$

For $y, z \in H$, we define the norm

$$\| y - z \|_H = \sup_{-r \leq t \leq T} \| y(t) - z(t) \| .$$

Then H is a complete normed space. since $T(t)$ is strongly continuous and ϕ is continuous, let

$$\| \phi(t+\theta) - \phi(\theta) \|_{D(B)} < \gamma/3 \text{ and}$$
$$\| T(t)B\phi(0) - B\phi(0) \| < \gamma/3.$$

We define

$$(Gy)(t) = \begin{cases} T(t)B\phi(0) + \int_0^t T(t-s)g(s, B^{-1}y_s, \int_0^s k(s,\tau,B^{-1}y_\tau)d\tau)ds, \\ B\phi(t) \quad -r \leq t \leq 0 \end{cases}$$

We claim that G maps H into H. In other word, for $y\in H$, we only show that $(Gy)_t\in H$. If $-r\leq t+\theta\leq 0$, then $(Gy)(t+\theta)=B\phi(t+\theta)$ an so

$$\| (Gy)_t - B\phi \|_C < \gamma/3. \tag{3.1}$$

If $0<t+\theta\leq T$, then

$$\begin{aligned}
&\| (Gy)(t+\theta) - B\phi(\theta) \| \\
&\quad \leq \| T(t+\theta)B\phi(0) - B\phi(0) \| \\
&\quad + \| B\phi(0) - B\phi(\theta) \| + \int_0^{t+\theta} \| T(t+\theta-s) \| \\
&\qquad \| g(s, B^{-1}y_s, \int_0^s k(s,\tau,B^{-1}y_\tau)d\tau \| ds \\
&\quad \leq \gamma/3 + \gamma/3 \\
&\quad + Me^{w}(t+\theta)\int_0^{t+\theta} e^{-ws} \|g(s, B^{-1}y_s, \int_0^s \\
&\qquad k(s,\tau,B^{-1}y_\tau)d\tau) - g(s,\phi,\int_0^s k(s,\tau,\phi)d\tau) \|ds \\
&\quad + Me^{w(t+\theta)} \int_0^{(t+\theta)} e^{-ws} \|g(s,\phi,\int_0^s k(s,\tau,\phi)d\tau \|ds \\
&\quad \leq \gamma/3 + \gamma/3 + Me^{wt} \int_0^t L_1(s)e^{-ws}[\| y_s - B\phi \|_C \\
&\quad + \int_0^s L_2(\tau)e^{w(s-\tau)} \| y_\tau - B\phi \|_C d\tau]ds \\
&\quad + Me^{wt} \int_0^t L_1(s)e^{-ws}[\| B\phi \|_C \\
&\quad + \int_0^s L_2(\tau)e^{w(s-\tau)} \| B\phi \|_C d\tau]ds \\
&\quad \leq \gamma/3 + \gamma/3 + MK_1Te^{wT}[1+K_2T][\gamma + \| \phi \|_{C(B)}] \\
&\quad \leq \gamma,
\end{aligned}$$

where K_1, K_2 are integral values of $L_1, L_2\in C(R^+, R^+)$, respectively. So $\| (Gy)t - B\phi \|_C<\gamma$. By(3.1) and (3.2), we obtain that $\| (Gy)_t - B\phi \|_C<\gamma$. Thus$(Gy)_t\in H$. Now, we prove that G is a Contraction from H to H. For $y, z\in H$, $t\geq 0$,

$$\begin{aligned}
&\| (Gy)(t) - (Gz)(t) \| \\
&\quad \leq \int_0^t \|T(t-s)\| \| g(s,B^{-1}y_s, \int_0^s k(s,\tau,B^{-1}y_\tau)d\tau) \\
&\qquad - g(s,B^{-1}z_s, \int_0^s k(s,\tau,B^{-1}z_\tau)d\tau \|ds \\
&\quad \leq Me^{wt} \int_0^t L_1(s)e^{-ws}[\| y_s - z_s \|_C \\
&\quad + \int_0^s L_2(\tau)e^{w(s-\tau)} \| y_\tau - z_\tau \|_C d\tau]ds \\
&\quad \leq Me^{wt} \int_0^t L_1(s)e^{-ws}[1+\int_0^s L_2(r)e^{w(s-\tau)} \|d\tau] \\
&\qquad ds \| y-z \|_H \\
&\quad \leq MK_1T[1+K_2T]e^{wT} \| y-z \|_H \\
&\quad < \| y-z \|_H
\end{aligned}$$

So

$$\|Gy - Gz\|_H = \sup_{-r\leq t\leq T} \|(Gy)(t) - (Gz)(t)\| < \| y-z \|_H.$$

Therefore, by Contraction mapping theorem, there is a unique $y\in H$ such that $Gy=y$. Since $x(t)=B^{-1}y(t)$, $x(t)$ is the unique solution of (1.1).

Theorem 3.2.

It the assumptions of Theorem 3.1 hold, then, for $t\in[0, T]$, $\phi, \bar{\phi}\in C(B)$,

$$\|x_t(\phi) - x_t(\bar{\phi})\|_{C(B)} \leq M\|\phi - \bar{\phi}\|_{C(B)}[1+K_1Te^{(K_1+K_2)T}]e^{wt}$$

Proof. Let $y(t)=Bx(\phi)(t)$, $\bar{y}(t)=Bx(\bar{\phi})(t)$, where $\phi, \bar{\phi}\in C(B)$ and $x(\phi)$, $x(\bar{\phi})$ be solutions corresponding to ϕ and $\bar{\phi}$, respectively. for $t+\theta\geq 0$

$$\begin{aligned}
&\| y(t+\theta) - \bar{y}(t+\theta) \| \\
&\quad \leq \| T(t+\theta) \| \| B\phi(0) - B\bar{\phi}(0) \| + \int_0^{t+\theta} \| \\
&\qquad T(t+\theta-s)\|L_1(s)\cdot[\|y_s - \bar{y}_s\|_C + \int_0^s L_2(\tau)e^{w(s-\tau)} \\
&\qquad \| y_\tau - \bar{y}_\tau \|_C d\tau]ds \leq Me^{wt}\|\phi - \bar{\phi}\|_{C(B)} + Me^{wt} \\
&\qquad \int_0^t L_1(s)e^{-ws} \| y_s - \bar{y}_s \|_C ds + Me^{wt} \int_0^t L_1(s) \\
&\qquad \int_0^s L_2(\tau)e^{-w\tau} \|y_\tau - \bar{y}_\tau \|_C d\tau ds
\end{aligned}$$

So.

$$\begin{aligned}
&e^{-wt} \| y_t - \bar{y}_t \|_C \\
&\quad \leq M \|\phi - \bar{\phi}\|_{C(B)} + M\int_0^t L_1(s)e^{-ws} \| y_s - \bar{y}_s \|_C ds \\
&\quad + M\int_0^t L_1(s) \int_0^s L_2(\tau)e^{-w\tau} \| y_\tau - \bar{y}_\tau \|_C d\tau ds
\end{aligned}$$

By Lemma 2.3,

$$\begin{aligned}
&e^{-wt} \| y_t - \bar{y}_t \|_C \\
&\quad \leq M\|\phi - \bar{\phi}\|_{C(B)} + [1+\int_0^t L_1(s)\exp[\int_0^s (L_1(\tau) \\
&\quad + L_2(\tau))d\tau]ds \\
&\quad \leq M \|\phi - \bar{\phi}\|_{C(B)}[1+K_1Te^{(K_1+K_2)T}]
\end{aligned}$$

So,

$$\|y_t - \bar{y}_t\|_C \leq M \|\phi - \bar{\phi}\|_{C(B)}[1+K_1Te^{(K_1+K_2)T}]e^{wt} \tag{3.3}$$

For $-r\leq t+\theta\leq 0$, this is clear, i.e.,

$$\| y_t - \bar{y}_t \|_C \leq \| \phi - \bar{\phi}\|_{C(B)} \tag{3.4}$$

Therefore, by (3.3) and (3.4)

$$\| y_t - \bar{y}_t \|_C \leq M \| \phi - \bar{\phi} \|_{C(B)}[1+K_1Te^{(K_1+K_2)T}]e^{wt}$$

Hence

$$\begin{aligned}
&\| x_t(\phi) - x_t(\bar{\phi}) \|_{C(B)} \\
&\quad \leq M \|\phi - \bar{\phi}\|_{C(B)}[1+K_1Te^{(K_1+K_2)T}]e^{wt}
\end{aligned}$$

In next Theorem, we show that the solution $x(t)$ of (1.1) is exponentially asymptotically stable. So we take(C_{22}) and (C_{42}) to conditions for semigroup $T(t)$ and function k.

Theorem 3.3.

Suppose assmptions (C_1), (C_{22}), (C_3), (C_{42}) hold, then every solution $x(\phi)(t)$ satisfies $\lim_{t\to\infty} \| x_t \|_{C(B)}=0$.

Proof. For $t+\theta\le 0$,

$$\| y(t+\theta)\le \| T(t+\theta) \| \| B\phi(0) \| + \int_0^{t+\theta} \| T(t+\theta-s) \| \| g(s, B^{-1}y_s, \int_0^s k(s,\tau,B^{-1}y_\tau)d\tau \| ds \le Me^{-\omega(t+\theta)} \| \phi \|_{C(B)} + \int_0^{t+\theta} Me^{-\omega(t+\theta-s)} L_1(s)\cdot [\| B^{-1}y_s \|_{C(B)} + \int_0^s L_2(\tau)e^{-\omega(s-\tau)} \| B^{-1}y_\tau \|_{C(B)} d\tau]ds$$

Then

$$e^{\omega t} \| y_t \|_C \le Me^{\omega r} \| \phi \|_{C(B)} + \int_0^t Me^{\omega r} L_1(s)e^{\omega s} \| y_s \|_C ds + \int_0^t ML_1(s)e^{\omega r} \int_0^s L_2(\tau)e^{\omega\tau} \| y_\tau \|_C d\tau ds$$

Put $p(t)=e^{\omega t}\| y_t \|_C$, by Lemma 2.3

$$\| y_t \|_C \le Me^{\omega r} \| \phi \|_{C(B)} \{1+\int_0^t Me^{\omega r}L_1(s) \cdot \exp[\int_0^s (ML_1(\tau)e^{\omega r}+L_2(\tau))d\tau]ds\} \cdot e^{-\omega t}$$

Therefore $\| y_t \|_C \le N \| \phi \|_{C(B)} e^{-\omega t}$, where N is constant. i.e.,

$$\lim_{t\to\infty} \| x_t \|_{C(B)} - \lim_{t\to\infty} \| y_t \|_C = 0.$$

4. Application

In order to illustrate the applications of our theorem established in previous sections, we consider the following partial functional integrodifferential equation ;

$$\frac{\partial}{\partial t}(z(x,t)-z_{xx}(x,t))-z_{xx}(x,t)=h(t,z(x,t-r), \int_0^t f(t,s,z(x,s-r))ds), \quad 0\le x\le\pi,\ t\in J_0 \qquad (4.1)$$

$$z(0,t)=z(\pi,t)=0,\ t\in J_0$$

$$z(x,t)=\phi(x,t),\ 0\le x\le\pi,\ -r\le t\le 0.$$

where $h: J_0\times R\times R\to R$, $f: J_0\times J_0\times R\to R$ are Lipschitz continuous functions with Lipschitz constants σ_1, σ_2, respectively. And $h(t,0,0)=0$, $f(t,s,0)=0$. Let $X=Y=L^2(0,\pi)$ and A, $B: X\to Y$ are operators defined by $Au=-u''$ and $Bu=u-u''$, $D(A)=D(B)=\{u\in X \mid u, u'$ are absolutely continuous, $u''\in X$, $u(0)=u(\pi)=0\}$. We now define mapping $H: J_0\times J_0\times C\to X$ and $F: J_0\times C\times X\to X$ as follows ;

$$H(t,\phi,y)(x)=h(t,\phi(-r)(x),y(x))$$
$$F(t,s,\phi)(x)=f(t,s,\phi(-r)(x))$$

Then Eq.(4.1) can be formulated abstractly as

$$(Bu(t))'+Au(t)=H(t,u_t,\int_0^t F(t,s,u_s)ds),\ t\in[0,T]$$
$$u(t)=\phi(t),\ -r\le t\le 0,\ \phi\in D(B).$$

And

$$Au=\Sigma_{n=1}^{\infty} n^2(u,v_n)v_n,\ u\in D(A)$$
$$Bu=\Sigma_{n=1}^{\infty} 1(1+n^2)(u,v_n)v_n,\ u\in D(B)$$

where $\{v_n\}_{n=1}^{\infty}$ is a complete set of othonormal eigenvectors of A with $v_n(x)=(2/\pi)^{1/2}\sin nx$. If $u\in X$, we obtains

$$B^{-1}u=\Sigma_{n=1}^{\infty} 1/(1+n^2)(u,u_n)u_n,$$
$$-AB^{-1}u=\Sigma_{n=1}^{\infty} -n^2/(1+n^2)(u,u_n)u_n,$$
$$T(t)u=\Sigma_{n=1}^{\infty} e^{-(n^2/(1+n^2))t}(u,u_n)u_n.$$

Then $-AB^{-1}$ is a bounded linear operator from X to X and $\| T(t) \| \le e^{-t}$ for all $t\ge 0$. Finally, we show that H satisfies conditions(C_3), (C_{41}).

$$\begin{aligned}
&\| H(t,\phi,y)-H(t,\bar\phi,y) \|^2 \\
&\le \int_0^\pi | h(t,\phi(-r)(x),y(x))-h(t,\bar\phi(-r)(x),y(x)) |^2 dx \\
&\le \int_0^\pi [\sigma_1 | \phi(-r)(x)-\bar\phi(-r)(x) |]^2 dx \\
&\le \sigma_1^2 \int_0^\pi | \phi(-r)(x)-\bar\phi(-r)(x) |^2 dx \\
&\le \sigma_1^2 \int_0^\pi | \phi-(r)(x)-\bar\phi(-r)(x) |^2 dx \\
&= \sigma_1^2 \Sigma_{n=1}^{\infty} (\phi(-r)-\bar\phi(-r), v_n)^2 \\
&\le \sigma_1^2 \Sigma_{n=1}^{\infty} (1+n^2)(\bar\phi(-r)-\bar\phi(-r), v_n)^2 \\
&\le \sigma_1^2 \| B(\phi(-r)-\bar\phi(-r) \|^2 \\
&= \sigma_1^2 \| \phi(-r)-\bar\phi(-r) \|_{D(B)}^2
\end{aligned}$$

$$=\sigma_1^2 \| \phi - \bar{\phi} \|_{D(B)}^2$$

Similarly,

$$\| F(t, s, \phi) - F(t, s, \bar{\phi}) \|^2 \le \sigma_2^2 \| \phi - \bar{\phi} \|_{C(B)}^2$$

Therefore we can apply Theorem 3.1 to Eq.(4.1).

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