LINEAR OPERATORS STRONGLY PRESERVING MULTIVARIATE MAJORIZATION

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ABSTRACT. In this paper, we will investigate the set of linear operators that strongly preserve multivariate majorization. We determine the linear operators that strongly preserve multivariate majorization with $T(I) = I$ and which map nonnegative matrices to nonnegative matrices.

1. Introduction

Our interest is the subject of majorization for matrices. We generalize the definition of majorization from vectors to matrices. It is called *"multivariate majorization"*. This basic idea makes sense whether the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ are points on the real line or points in a more general linear space. Very little is known about majorization where the components of $\boldsymbol{a}$ and $\boldsymbol{b}$ are not in $\boldsymbol{R}^n$ ([1],[5]).

Let $\mathcal{A}$ be a linear space of matrices, T be a linear operator on $\mathcal{A}$, and $\mathcal{R}$ be a relation on $\mathcal{A}$. A linear operator T is called *strongly preserves* $\mathcal{R}$ if

$$\mathcal{R}(T(X), T(Y)) \quad \text{if and only if} \quad \mathcal{R}(X, Y).$$

Those linear operators on a matrix space that preserve commuting pairs of matrices were characterized in [2]. In 1987, all similarity preserving operators on the $n \times n$ complex matrices, unitary equivalence preserving linear operators on the Hermitian matrices, and (sub)majorization preserving linear operators on Hermitian matrices was determined in [3]. And characterizations of linear operators on a matrix space that preserve consimilarity, $*$-congruence, nonsingularity, and unitary equivalence were obtained in [4].

In this paper, we will study linear operators that strongly preserve multivariate majorization. For a simple characterization, we need the hypothesis that $T(I)$ is equal to I. We determine the linear operators that strongly preserve multivariate majorization with $T(I) = I$ and such that T preserves the set of nonnegative matrices.

1991 *Mathematics Subject Classification.* 15A04, 15A21, 15A30.
Key words and phrases. linear operators, multivariate majorization.

2. Main results

A nonnegative real matrix is called *doubly stochastic* if each of its row sums and column sums are equal to 1. Let Ω_n denote a set of all $n \times n$ doubly stochastic matrices. The set Ω_n is closed under matrix multiplcations, conjugate transposition, and convex combinations, that is, if A, B are doubly stochastic, so are AB, A^*, and $\alpha A + (1-\alpha)B$ for all $0 \leq \alpha \leq 1$. An obvious example of a doubly stochastic matrix is the $n \times n$ matrix in which each entry is $\frac{1}{n}$, J_n. It was showed that this is the unique irreducible idempotent $n \times n$ doubly stochastic matrix in [6]. In [7], it was showed that if P and $P^{-1} = Q$ are both doubly stochastic, then P is a permutation matrix. Let $M_n(\boldsymbol{R})$ denote the set of all $n \times n$ matrices over the real field $\boldsymbol{R}$.

DEFINITION 2.1. *Let A and B be $n \times n$ real matrices. Then A is said to be multivariate majorized by B, written $A \prec^{mul} B$, if there exists an $n \times n$ doubly stochastic matrix D such that $A = BD$.*

THEOREM 2.2. *Let T be a linear operator on $M_n(\boldsymbol{R})$ that strongly preserves multivariate majorization, then T is nonsingular.*

Proof. Suppose $T(X) = O$. Then $T(X) \prec^{mul} O$. Since T is linear, $T(O) = O$. This implies

$$X \prec^{mul} O$$

beacuse T strongly preserves multivariate majorization. By the definition of multivariate majorization, there exists an $n \times n$ doubly stochastic matrix D such that $X = O \cdot D$. Hence $X = O$. ■

Now, we will find an interesting property of a multivariate majorization strong preserver.

THEOREM 2.3. *Let T be a linear operator on $M_n(\boldsymbol{R})$ that strongly preserves multivariate majorization. Then the followings are equivalent:*

(1) *$T(P) = Q$ where P and Q are permutation matrices;*
(2) *$T(D) = S$ for D, $S \in \Omega_n$;*
(3) *$T(J_n) = J_n$.*

Proof. (1) $\Leftrightarrow$ (2): For any doubly stochastic matrix D, we have $D \prec^{mul} P$ for every permutation matrix P. Hence

$$T(D) \prec^{mul} T(P) = Q$$

where Q is a permutation matrix. Therefore $T(D) \in \Omega_n$. The converse is similar.
(2) $\Leftrightarrow$ (3): Since $J_n \prec^{mul} D$ for any doubly stochastic matrix D,

$$T(J_n) \prec^{mul} T(D) = S$$

for $S \in \Omega_n$. Therefore $T(J_n) = J_n$. Converse is obtained similarly. ∎

LEMMA 2.4. *[7] If D and D^{-1} are in Ω_n, then D is a permutation matrix.*

COROLLARY 2.5. *Let T be a linear operator on $M_n(\boldsymbol{R})$ that strongly preserves multivariate majorization. If $T(I) = I$, then the three conditions of Theorem 2.3 hold.*

Proof. It is sufficient to show that $T(P) = Q$ for permutation matrices P and Q. For any permutation matrix P,

$$P \prec^{mul} I \prec^{mul} P.$$

So, there exist doubly stochastic matrices D and S such that

$$T(P) = I \cdot D = D$$

and

$$I = T(P) \cdot S.$$

This implies $I = D \cdot S$. Thus,

$$T(P) = Q$$

where Q is a permutation matrix by Lemma2.4. The proof is complete. ∎

Let E_{ij} denote the matrix with a 1 in the (i, j) entry and zero in every other entry and $M_n(\boldsymbol{R}^+)$ denote the set of all $n \times n$ nonnegative matrices over the real field $\boldsymbol{R}$.

Theorem 2.6. *Let T be a linear operator on $M_n(\boldsymbol{R}^+)$ that strongly preserves multivariate majorization. If $T(I) = I$, then there exists a permutation matrix P such that $T(X) = P^T X P$ for every $X \in M_n(\boldsymbol{R}^+)$.*

Proof Since $T : M_n(\boldsymbol{R}^+) \to M_n(\boldsymbol{R}^+)$, we must have that $T(E_{ii})$ has only nonzero entries on the main diagonal since $T(I) = I$. But, by Corollary 2.5, $T(E_{ii} + Q)$ is a permutation matrix for every matrix Q such that $E_{ii} + Q$ is a permutation matrix. Since T is nonsingular, it follows that $T(E_{ii}) = E_{jj}$ for some j. Let P be the permutation matrix such that $PT(E_{ii})P^T = E_{ii}$ for every i. Such a permutation matrix exists since T is nonsingular. Let $T_1(X) = PT(X)P^T$ for all X.

Now, by an argument similar to the above, using a permutation matrix that has a 1 in the (i, j) entry, we have that $T_1(E_{ij}) = E_{rs}$ for some (r, s). If $r \neq i, j$ and $s \neq i, j$ then there is a permutation matrix, S, which has a 1 in the (i, i) entry and a 1 in the (r, s) entry, but then ${T_1}^{-1}(S)$ must be a permutation matrix which has 1's in the (i, i) and (i, j) entries, an impossibility. Thus, suppose $r = i$ and $i \neq j$. Then, if $s \neq j$, there is a permutation matrix, R, which has 1's in the (i, s) and (j, j) entries. But then, ${T_1}^{-1}(R)$ is a permutation matrix with 1's in the (i, j) and (j, j) entries, also an impossibility. Thus $T_1(E_{ij}) = E_{ij}$ or $T_1(E_{ij}) = E_{ji}$. Further, since T and hence

T_1 preserves permutation matrices, $T_1(E_{ij}) = E_{ij}$ for all (i,j) or $T_1(E_{ij}) = E_{ji}$ for all (i,j). Thus, we have that $T_1(X) = X$ for all X or $T_1(X) = X^T$ for all X. Now, let K be the matrix with 1's in each entry of the first column, and zeros elsewhere. Then, K majorizes J_n, but K^T does not, thus, the map $X \rightarrow X^T$ does not preserve multivariate majorization. That is T_1 is the identity transformation, and it follows that $T(X) = P^T X P$ for all X. ■

REFERENCES

[1]. T. Ando, Majorization and inequalities in matrix theory, *Linear Algebra Appl.* **199** (1994), 17–67.

[2]. L. B. Beasley, Linear transformations on matrices: The invariance of commuting pairs of matrices, *Linear and Multilinear Algebra* **6** (1978), 179–183.

[3]. F. Hiai, Similarity preserving linear maps on matrices, *Linear Algebra Appl.* **97** (1987), 127–139.

[4]. R. A. Horn, C. K. Li and N. K. Tsing, Linear operators preserving certain equivalence relations on matrices, *SIAM J. Matrix Anal. Appl.* **12** (1991), 195–204.

[5]. A. W. Mashall and I. Olkin, *Inequalities: Theory of Majorization and Its applications*, Academic, New York, 1972.

[6]. S. Schwarz, A note on the structure of the semigroup of doubly stochastic matrices, *Mat. Casopis Sloven. Akad. Vied.* **17** (1967), 308–316.

[7]. T. Snijders, *An ordering of probability distributions on a partially ordered outcome space*, Department of Math., Univ. of Groningen, The Netherlands, 1976.

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