On Cf-semistratifiable Spaces

劉　　成　　龍

cf-semistratifiable한 空間에 관하여

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要　約

1969年에 D. J. Lutzer가 K-semistratifiable 한 空間을 紹介했고 이듬해인 1970年에 G. D. Creede가 "semistratifiable 한 空間에 관한 硏究"를 發表했다. 本 論文에서는 이들 空間사이에 놓인 cf-semistratifiable 한 空間을 定義하고 諸般性質을 考察하고자 한다. 即,

모든 cf-semistratifiable 한 空間은 σ-cushioned cf-pairnet의 性質을 가지며 有限個의 閉 cf-semistratifiable 한 空間의 合도 cf-semistratifiable 한 空間이며 閉連續 pseudo open 寫像에 依한 cf-semistratifiable한 空間의 像도 또한 cf-semistratifiable 한 空間이다.

다음에 主要한 結果로서 cf-semistratifiable 한 空間은 G_δ^*-diagonal을 가져서 developable 한 空間이 된다. 이때, 여기에 "regular"인 條件만 주면 모든 可算個의 compact cf-semistratifiable 空間은 metrizable함을 밝힌다.

Abstract

In this paper cf-semistratifiable space is introduced and the relationships between cf-semistratifiable spaces and several classes of topological spaces are investigated. It is shown that a locally finite union of closed cf-semistratifiable space is cf-semistratifiable

and the image of cf-semistratifiable space under a closed continuous pseudo open map is cf-semistratifiable. And we give some conditions for cf-semistratifiable spaces to be α-spaces.

As the main result we prove that every regular cf-semistratifiable space has a G_δ^*-diagonal and a space X is a cf-semistratifiable $w\Delta$-space if and only if it is developable. This result is used to show that in $w\Delta$-space, every compact cf-semistratifiable space is metrizable.

1. Introduction

In [1], J.G. Ceder defined an M_3-space to be a T_1-space with a σ-cushioned pairbase (see definition in §1. below) as a generalization of metric spaces, M_3-space was renamed stratifiable space by Borges [8]. G.D. Creede [2] again generalized the space to semistratifiable spaces which lies between the class of semi-metric spaces and the class of spaces in which closed sets are G_δ.

In this paper we study a class of spaces called cf (convergent filterbase)-semistratifiable spaces which lies between the semistratifiable spaces and K-semistratifiable spaces. We wish to discuss a means of constructing topological spaces which may deserve to be better known. We begin with a slight modification due to Creede [2] and Sakong [21].

Through out this paper the set of positive integers will be denoted by N. Most terms and notations which are not defined in this paper are used as in J. Dugundji [14]. In what follows, all spaces are assumed to be T_1.

Let F be a map from $N\times\mathscr{T}$ to the family of all closed subsets of a spaces $(X, \mathscr{T})$. Consider the following conditions of F;

(a) For each $U\in\mathscr{T}$, $U=\bigcup_{n=1}^{\infty} F(n, U)$

(b) If U, $V\in\mathscr{T}$ and $U\subset V$, then $F(n, U)\subset F(n, V)$ for each $n\in N$.

(c) For each convergent filterbase $\mathscr{U}=\{A_\alpha: \alpha\in\mathscr{A}$, where $\mathscr{A}$ is a directed set$\}$ to x in X, and for $U\in\mathscr{T}$, containing x, there exist a $k\in N$ and a $\beta\in\mathscr{A}$ such that $x\in F(k, U)$ and $A_\alpha\subset F(k, U)$ for all $\alpha\geq\beta$, $\alpha\in\mathscr{A}$.

(d) For each $U\in\mathscr{T}$, $U=\bigcup_{n=1}^{\infty} F(n, U)^0$ where $F(n, U)^0$ is the interior of $F(n, U)$. In [8] F is called a stratification for X if F satisfies (a), (b) and (d).

(e) If $C\subset U$ with C compact and U open there is a $n\in N$ such that $C\subset F(n, U)$.

In [5] F is called a K-semistratification for X if it satisfies (a), (b) and (e) and finally a semistratification whenever it satisfies (a) and (b) in [2].

Definition 1.1.

A map F which satisfy (a), (b) and (c) is called a cf-semistratification for X. A topological space is said to be a cf-semistratifiable space whenever it has a cf-semistratification.

By comparing the above definition with Definition 1.1. of [8]. One can seethat stratifiable→K-semistratifiable→cf-semistratifiable→semistratifiable.

In section 2, stratifiable spaces and semistratifiable spaces being initiated by Borges [8] and Creede [2] respectively, we verify some relationships of their several similarities which are given by means of a σ-cushioned pair-filterbase and cf-semistratifiable functions.

In section 3, some properties of cf-semistratifiable spaces are shown. Moreover we note that cf-semistratifiability is hereditary.

In section 4, we show that the image of a cf-semistratifiable space under a closed continuous pseudo-open map is cf-semistratifiable.

In section 5, the cf-stratifiable $w\Delta$-spaces are shown to be semidevelopable, every(regular) cf-semistratifiable space has a G_δ^*-diagonal, and finally a space X is a cf-semistratifiable $w\Delta$-space if and only if it is developable.

In the last section 6, we give necessary and sufficient conditions for a cf-semistratifiable space to be metrizable.

2. Relationships between the cf-semistratifiable space and other topological spaces

A net work (or net) [16] in a space X is collection $\mathscr{B}$ of subsets of X such that given any open subset $U \subset X$ and $x \in U$, there is a member B of $\mathscr{B}$ such that $x \in B \subset U$. A K-net (called a pseudo base by Michael in [24]) is a collection $\mathscr{B}$ of subsets of X such that given any compact subset K and any open subset U of X containing K, there is a $B \in \mathscr{B}$ such that $K \subset B \subset U$. A cs-network [25] is a collection $\mathscr{B}$ of subsets of X such that given any convergent sequence $x_n \to x$ and any open U containing x, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$ and $\langle x_n \rangle$ is eventually in B.

By peering into the above definitions, one can see that any K-network is a cs-network, which is a network. A space with a σ-locally finite network is called a σ-space and a regular space with a countable network is called a cosmic space [24] (In[3], X is a cosmic space if and only if it is the regular continuous image of a separable metric space).

Let $\mathscr{P}$ be a collection of ordered pairs of subsets of the T_1-space X such that, for each $P=(A, B)\in\mathscr{P}$. A is open and $A\subset B$, and such that, for every $x\in X$ and every neighborhood U of x, there is a $P\in\mathscr{P}$ for which $x\in A\subset B\subset U$. Then $\mathscr{P}$ is called a pair base for X. Moreover, $\mathscr{P}$ is called cushioned if, for every $Q\subset\mathscr{P}$, $Cl\cup\{A;\ P=(A;B)\in Q\}\subset\cup\{B;\ P\in Q\}$ and $\mathscr{P}$ is σ-cushioned if it is the union of countably many cushioned collections. A M_3-space is a T_1-space with a σ-cushioned pair base. These definitions are due to R.W. Heath in [3]. Note that Borges calls M_3-spaces stratifiable spaces in [8]. According to this, M_3-spaces are cf-stratifiable spaces.

We are about to generalize the concepts of network and pairbase. Let $\mathscr{P}$ be a collection of ordered pairs $P=(A, B)$ of subsets of the space X with $A\subset B$ for all $P\in\mathscr{P}$(Here, A is not necessarily open). Then $\mathscr{P}$ is called a pairnet for X if for every $x\in X$ and every neighborhood U of x, there is a $P\in\mathscr{P}$ for which $x\in A\subset B\subset U$. Moreover, $\mathscr{P}$ is said to be cushioned if for every $Q\subset\mathscr{P}$, $Cl\cup\{A;\ P=(A, B)\in Q\}\subset\cup\{B;\ P\in Q\}$, $\mathscr{P}$ is said to be σ-cushioned if it is a union of countably many cushioned collections.

A pair net is called a K-pairnet if given any $C\subset U$ with C compact and U open, there is a $P\in\mathscr{P}$ for which $C\subset A\subset B\subset U$.

A pairnet is called a cf-pairfilterbase if given any convergent filterbase $\mathscr{U}=\{A_\alpha;\ \alpha\in\mathscr{A}\}$ converging to x and an open subset U containing x, there exist a $P\in\mathscr{P}$ and a $\beta\in\mathscr{A}$ for which $x\in A_\alpha\subset A\subset B\subset U$, for all $\alpha\geqq\beta$, $\alpha\in\mathscr{A}$.

It is easily checked that any K-pairnet is cf-pairfilterbase and a cf-pairfilterbase is a pairnet. Note that Borges calls M_3-spaces stratifiable spaces in [8], which was charaterized as;

A T_1-spaces is stratifiable if and only if there is a map T: $N\times\mathscr{T}\to\mathscr{T}$ such that,

(a) For each $U\in\mathscr{T}$, $Cl\ T(n, U)\subset U$

(b) For each $U\in\mathscr{T}$, $\bigcup_{n=1}^{\infty} T(n, U)=U$, and

(c) For U, $V\in\mathscr{T}$, $T(n, U)\subset T(n, V)$ whenever $U\subset V$.

We can see that the above characterization is equivalent to the definition of Ceders' in [1].

As for the stratifiable spaces, semistratifiable spaces can be charaterizedas follows;

Theorem 2.1. For a topological space $(X, \mathscr{T})$, the followings are equivalent.

(1) X is semistratifiable

(2) X has a σ-cushioned pairnet

(3) there is a function

g: $N\times X\to\mathscr{T}$ such that

(a) $\bigcap_{i=1}^{\infty} g_i(x)=\{x\}$ for each $x \in X$ and

(b) if y is a point of $g_i(x_i)$ forall $i \in N$. Then the sequence $\langle x_n \rangle$ which lies in X converges to y.

Proof. (1)⇔(3) follows from the theorem 1.2 due to Creede [2]. To show(1)⇔(2), let F be a semistratification of X. Define $\mathscr{P}_n=\{(F(n, U), U): U \in \mathscr{T}\}$ for each $n \in N$. Then each $\mathscr{P}_n$ is cushioned. Since, let $\mathscr{T}'$ be a subcollection of $\mathscr{T}$, then $F(n, U) \subset F(n, \bigcup\{U: U \in \mathscr{T}'\})$ for each $U \in \mathscr{T}'$, and so $\bigcup\{U: U \in \mathscr{T}'\} \supset F(n, \bigcup\{U: U \in \mathscr{T}'\}) = Cl\, F(n, \bigcup\{U: U \in \mathscr{T}'\}) \supset Cl \bigcup\{F(n, U): U \in \mathscr{T}'\}$ Conversely, suppose there is a σ-cushioned pairnet. $\mathscr{P}=\bigcup \mathscr{P}_n$ for X. For each $n \in N$ and $U \in \mathscr{T}$, $F(n, U)=Cl \bigcup\{A: P=(A, B) \in \mathscr{P}_n, B \subset U\}$. It is clear that F is a semistratification for X.

For K-semistratifiable spaces we have an analogous result whose proof can be proved by taking an analogous process to theorem 2.1 and so we can omit the proof of the following theorem 2.2.

Theorem 2.2. For a space $(X, \mathscr{T})$, the followings are equivalent;

(1) X is K-semistratifiable

(2) X has a σ-cushioned K-pairnet

(3) there is a semistratifiable function g with an additional condition:

(c) if $C \subset U$ with C compact and U open in X, then there is an $n \in N$ with $C \cap (\bigcup\{g_n(x): x \in X-U\})=\emptyset$. In this case, g is called a K-semistratifiable function.

Theorem 2.3. For a space $(x, \mathscr{T})$, the followings are equivalent.

(1) X is cf-semistratifiable

(2) X has a σ-cushioned cf-pairfilterbase

(3) There is a semistratifiable function g with an additional condition;

(d) Given a convergent filterbase $\mathscr{U}=\{A_\alpha: \alpha \in \mathscr{A}\}$ to x in X and an open set $U \subset X$ containing x, there is a $K \in N$ such that $A_\alpha \not\subset \bigcup_{x \in X-U} g_k(x)$ and $\{\alpha \in \mathscr{A}; A_\alpha \subset \bigcup_{x \in X-U} g_k(x)\}$ is finite. In this case g is called a cf-semistratifiable function.

Proof: (1)⇔(2) follows from theorem 2.1. To show that (1)⇔(3), let F be a cf-semistratification for X. For each $n \in N$ and $x \in N$, define the function $g_n$by $g_n(x)=X-F(n, X-\{x\})$. Creede proved g is a semistratifiable function in [2]. To show g satisfies(d), consider the following $\bigcup_{x \notin V} g_k(x)=\bigcup_{x \notin V}\{X-F(k, X-\{x\})\}=X-\bigcap_{x \notin V} F(k, X-\{x\})$ which is contained in $X-F(k, V)$. If $A_\alpha \in \mathscr{U}$ is contained in $F(k, V)$ for all $\alpha \geqq \beta$, α, $\beta \in \mathscr{A}$, $\{\alpha \in \mathscr{A}: A_\alpha \subset \bigcup_{x \notin V} g_k(x)\}$ is finite. Conversely, by putting $F(n, U)=X-\bigcup_{x \notin X-U} g_n(x)$, we get a cf-semistratifiable function.

3. Properties of cf-semistratifiable spaces

As we would expect, the class of cf-semistratifiable spaces shares certain nice properties with the more familiar classes of spaces mentioned above.

Theorem 3.1. Every subspace of a cf-semistratifiable space is cf-semistratifiable.

Proof. Straightforward

Theorem 3.2. The countable product of cf-semistratifiable spaces is cf-semistratifiable.

Proof. For each i, let X_i be a cf-semistratifiable space and g_i (theorem 2.3.(3)) be cf-semistratifiable function. Let $X=\Pi_{i=1}^{\infty} X_i$ and let π_i be the projection of X onto X_i. For each i, j and $x\in X$, let $h_{ij}(x)=g_{ij}(\pi_i(x))$ if $i\leq j$ and $h_{ij}(x)=X_i$ if $i>j$.

Now let $g_j(x)=\prod_{i=1}^{\infty} h_{ij}(x)$ for each j and x. The function g satisfies the conditions of cf-semistratifiable function for the space X.

To prove that g satisfies (d) of theorem 2.3. (3), let $\mathscr{U}=\{A_\alpha:\ \alpha\in\mathscr{A}\{$ be a filterbase converging to z and let $z\in U\in\mathscr{T}$. Take a open neighborhood V of z, $V=\Pi_{i\in I}\ V_i\times\prod_{i\in(\mathscr{A}-I)} X_i\subset U$, where I is a finite subset of $\mathscr{A}$. For each i, $\{\pi_i(A_\alpha):\ \alpha\in\mathscr{A}\}$ is a set sequence containing $\pi_i(z)$, and $\pi_i(V)$ is open in X_i and contains $\pi_i(z)$. There is a k_i such that $\{\alpha\in\mathscr{A}:\ \pi_i(A_\alpha)\subset\cup\{g_{ik_i}(s):\ s\in X_i-\pi_i(V)\}\}$ is finite for each $i\in I$. Let $k=\max\{k_i:\ i\in I\}$.

Then $A_\alpha\subset\bigcup_{x\in X-V} g_k(x)$ if and only it there is an $x\in X-V$ such that $A_\alpha\subset g_k(x)$ if and only if there is an $x\in X$ such that $\pi_i(x)\in X_i-\pi_i(V)$ for some $i\in I$ and $A_\alpha\subset g_k(x)$ if and only if there is an $x\in X$, such that $\pi_i(x)\in X_i-\pi_i(V)$ for some $i\in I$ and $\pi_i(A_\alpha)\subset g_{ik}(\pi_i(x))$. This implies $\pi_i(A_\alpha)\subset\cup\{g_{ik}(s):\ s\in X_i-\pi_i(V)\}$. Thus $\{\alpha\in\mathscr{A}:\ A_\alpha\subset\bigcup_{x\in X-V} g_k(x)\}$ is finite, this insures that $\{\alpha\in\mathscr{A}:\ A_\alpha\subset\bigcup_{x\in X-U} g_k(x)\}$ is finite since $V\subset U$.

Theorem 3.3. The union of two closed cf-semistratifiable spaces is cfsemistratifiable.

Proof. Let X be $X_\alpha\cup X_\beta$ where either X_α or X_β is also cf-semistratifiable space respectively. If $X_\alpha\cap X_\beta=\emptyset$, then $X=X_\alpha\cup X_\beta$ is a cf-semistratifiable since there is a cf-semistratification function for X_α or X_β. Suppose $X_\alpha\cap X_\beta\neq\emptyset$, now let $V_n=[V\cap(X_\alpha-X_\beta)]_n\cup(V\cap X_\alpha\cap X_\beta)_n\cup[V\cap(X_\beta-X_\alpha)]_n$ for V open in X, then the correspondence $V\rightarrow\{V_n\}_{n=1}^{\infty}$ is a cf-semistratification for X satisfying the conditions of Definition 1.1. in §1.

Corollary 3.4. The union of a locally finite, closed cf-semistratifiable spaces is cf-semistratifiable.

Proof. Let X be the union of a locally finite collection $\{X_\alpha:\ \alpha\in J\}$ of closed cf-semistratifiable spaces. For each $\alpha\in J$, let F be a cf-semistratification for X_α and $\mathscr{T}$ the topology of X.

Define F: $N\times\mathscr{T}\rightarrow$the collection of closed subsets of X by $F(n,\ U)=\bigcup_{\alpha\in J} F_\alpha(n,\ U\cap X_\alpha)$. Each $\{F_\alpha(n,\ U\cap X_\alpha)$: $\alpha\in J\}$ is locally finite, hence it has the closure preserving property. This insures each $F(n,\ U)$ is closed in X. To show that F is a cf-semistratification for X. For the condition (a) of Definition in § 1,

$\bigcup_{n=1}^{\infty} F(n,\ U)=\bigcup_{n=1}^{\infty}\bigcup_{\alpha\in J} F(n,\ U\cap X_\alpha)=\bigcup_{\alpha\in J}\bigcup_{n=1}^{\infty} F_\alpha(n,\ U\cap X_\alpha)=\bigcup_{\alpha\in J}(U\cap X_\alpha)=U$. For (b): this condition is clear. For (c): Let $\mathscr{U}=\{A_\alpha:\ \alpha\in\mathscr{A}\}$ be a filterbase in X converging to x. Given an open set U of X containing x, there is an open set V such that $x\in V\subset U$ and $\{\beta\in J;\ X_\beta\cap V\neq\emptyset\}$ is finite. We denote this finite subset of J by $I=\{1,\ 2,\ \cdots,\ k\}$. For each $i\in I$, if $\{A_\alpha:\ A_\alpha\subset X_i\}$ is infinite, x must be contained in X_i. The cf-semistratifiability of X_i insures that there is an n_i such that A_α is contained in $F_i(n_i,\ V\cap X_i)$ for all $\alpha\geqq\beta(\alpha,\ \beta\in\mathscr{A})$. Let $n_0=\max\{n_i:\ i\in I\}$. Then A_α is contained in $F(n_0,\ V)=\bigcup_{i\in I} F_i(n_0,\ V\cap X_i)$ for all $\alpha\geqq\beta$ since I is finite.

Corollary 3.5. Every space with the properties of paracompactness and locally cf-semistratification is cf-semistratifiable.

Proof. Let X be the space given, for every $x\in X$, there is an open neighborhood $U(x)$ of x such that $U(x)$ is cf-semistratifiable and there is a locally finite closed refinement $\{B_\beta:\ \beta\in\mathscr{A}\}$ of $\{U(x):\ x\in X\}$. Then each B_β is cf-semistratifiable by theorem 3.1, so that X is cf-semistratifiable by using theorem 3.3.

4. Mappings

Charles C. Alexander introduced the concept of pseudo map in [18].

Definition 4.1[18]. Let X and Y be topological spaces. Then a surjective map f from X onto Y is pseudo-open if and only if for each $y\in Y$ and each open neighborhood U of $f^{-1}(y)$ in X, $y\in \mathrm{Int}\ f(U)$.

Theorem 4.2. The image of cf-semistratifiable space under a closed continuous pseudo-open map is cf-semistratifiable.

Proof. Let f be a closed continuous pseudo-open map from a cf-semistratifiable space X onto a space Y. Let F be a cf-semistratification for X.

For each open V, containing a point y of Y and $n\in N$, let $S(n,\ V)=f(F(n,\ f^{-1}(V))$. By Definition 4.1. S is a semistratification for Y. Moreover it clearly satisfies the condition (c) of § 1. Since, let U be open neighborhood in X including $f^{-1}(V(y))$, then X has a filter base $\mathscr{U}=\{A_\alpha:\ \alpha\in\mathscr{A}\}$ converging to $f^{-1}(y)$ in U such that $y\in \mathrm{int}\ f(U)$. On the other hand, owing to the cf-semistratifiability of X, there exist a $n_0\in N$ and a $\beta\in\mathscr{A}$

such that A_α is contained in $F(n_0, f^{-1}(V(y)))$ for $\alpha\in\mathscr{A}$, $\alpha\geqq\beta$. That is, $y\in f(A_\alpha)\subset\text{int}(f[F(n_0, f^{-1}(V(y))])$. Thus $f(A_\alpha)$ is contained in $S(n_0, V)=f[F(n_0, f^{-1}(V(y))]$ for $\alpha\geqq\beta$, $\alpha, \beta\in\mathscr{A}$.

5. Spaces with a semi-development and spaces with a G_δ^*-diagonal

Let X be a set, $\mathscr{G}$ a cover of X, x an element of X. The star of x with respect to $\mathscr{G}$, denoted by $st(x, \mathscr{G})$, is the union of all elements of $\mathscr{G}$ containing x. The order of x with respect to $\mathscr{G}$, denoted by $\text{ord}(x, \mathscr{G})$, is the number of elements of $\mathscr{G}$ containing x.

Definition 5.1[12]. A development for a space X is a sequence $\Delta=\{\mathscr{G}_n: n=1, 2, \cdots\}$ of open cover of X such that $\{st(x, \mathscr{G}_n): n=1, 2, \cdots\}$ is a local base at for each $x\in X$. A space is developable if and only if there exists a development for the space.

Definition 5.2[18]. Let $\Delta=\{\mathscr{G}_n: n=1, 2, \cdots\}$ be a sequence of (not necessarily open) covers of a space X.

(1) Δ is a semi-development for X if and only if for each $x\in X$, $\{st(x, \mathscr{G}_n): n=1, 2, \cdots\}$ is a local system of neighborhoods at x.

(2) A semi-development Δ of X is a strong semi-development if and only if for each $M\subset X$ and $x\in\overline{M}$ there is a descending sequence $\{G_n: n=1, 2, \cdots\}$ such that $x\in G_n\in\mathscr{G}_n$ and $G_n\cap M\neq\emptyset$.

Lemma 5.3. Let $\Delta=\{\mathscr{G}_n: n=1, 2, \cdots\}$ be a semi-development for a T_0-space X: If $\{G_n: n=1, 2, \cdots\}$ is a sequence of sets such that $G_n\in\mathscr{G}_n$ for each n, then $\cap\{G_n: n=1, 2, \cdots\}$ contains at most one point.

Proof. There is a point x such that $x\in G_n\subset st(x, \mathscr{G}_n)$ for each n, since a semi-development for the topological space X is T_1-space and then it has a local base at $x\in X$. Hence $x\in\cap\{G_n: n=1, 2, \cdots\}\subset st(x, \mathscr{G}_n)$.

From the aid of Alexander [18], recall that a space X is semi-metrizable if and only if it is a semi-developable T_0-space. Moreover every T_0-semidevelopable space is T_1.

Definition 5.4[8]. A space X is a $w\Delta$-space if there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of X such that, for each x in X, if $x_n\in st(x, \mathscr{G}_n)$ for $n=1, 2, \cdots$ then the sequence $\langle x_n\rangle$ has a cluster point. Such a sequence of open covers is called a $w\Delta$-sequence for X.

Lemma 5.5. A cf-semistratifiable $w\Delta$-space is semi-developable.

Proof. Let F is a cf-semistratification for a space X, and let $\Delta=\{\mathscr{G}_n: n=1, 2, \cdots\}$ is a $w\Delta$-sequence for the space X.

Taking $st(x, \mathscr{G}_n)$ such that $x\in st(x, \mathscr{G}_n)\subset F(k, U)$. For arbitrary open U in X, there is an A_α belonging to a convergent filter base $\mathscr{U}=\{A_\alpha: \alpha\in\mathscr{A}\}$ such that $x\in A_\alpha\subset st(x, \mathscr{G}_n)\subset F(k, U)\subset U$. Thus $\{\mathscr{G}_n: n=1, 2, \cdots\}$ is a semi-development for X.

Definition 5.6[18]. A topological space X is semi-metric if there is a distance function d; $X\times X\to R$ satisfying conditions such that,

(1) $d(x, y)=d(y, x)\geqq 0$

(2) $d(x, y)=0$ if and only if $x=y$,

(3) $x\in\overline{M}$ if and only if $d(x, M)=\inf\{d(x, m): m\in M\}=0$.

Creede [2] proved that a T_1-space is a semi-metric space if and only if it is a countable semistratifiable space. We can see an analogous result for a cf-semistratifiable space, replacing a semistratifiable space by a cf-semistratifiable space.

And we can see, from the above Lemma 5.5, that a cf-semistratifiable $w\Delta$-space is a semi-metric space.

Thus, given a semi-development $\Delta=\{\mathscr{G}_n: n=1, 2, \cdots\}$ for a topological space X, we will let d_Δ denote the semi-metric on X defined from Δ as theorem 1.3 in Alesander [18]. Similarly, given a semi-metric d on X, we will let Δ_d denote the semi-development on X defined from d as Alexander [18]. Hence we can define a semimetric d on the cf-semistratifiable $w\Delta$-space.

Theorem 5.7. In a cf-semistratifiable $w\Delta$-space, every convergent sequence has a Cauchy subsequence.

Proof. Let F be a cf-semistratification on X and let $S=\{x_n: n=1, 2, \cdots\}$ be a sequence in X converging to the point $x\in X$, If $x_n=x$ for infinitely many n, then clearly we can define a Cauchy subsequence of S, Otherwise let $F(n, U)=\{x_n: n=1, 2, \cdots\}-\{x\}$.

Then $x\in U$ implies, since $U=\cup F(n, U)$ in the condition (a) of Definition 1.1, that there is a descending sequence of sets$\{G_n: n=1, 2, \cdots\}$ of the convergent filter base $\mathscr{U}=\{A_\alpha: \alpha\in\mathscr{A}\}$ to x such that for every open neighborhood U of x we can find a $n_0\in N$ and for each $n\geqq n_0$, $x\in G_n$ and $G_n\cap F(n, U)\neq\emptyset$ for which $G_n\subset A_{\alpha_{n_0}}\subset U$. We now define a subsequence of $\{x_n: n=1, 2, \cdots\}$ inductively. Choose $x_n\in G_n\cap F(n, U)$ and $n\geqq n_0$. Now observe that $G_n\cap F(n, U)$ is infinite. For suppose not; say $G_n\cap F(n, U)=\{a_1, a_2, \cdots, a_m\}$. Then for sufficiently large number n_0, diam $G_{n_0}<\inf$ diam $\{d(x, a_i): i=1, 2, \cdots, m\}$ (Since X is semimetrizable if and only if X is a semidevelopable T_0-space and then every T_0 semi-developable space is T_1. By Creeds [2], T_1 is a semi-metric space if and only if it is a countable cf-semistratifiable space. Hence there is semi-metrie d on the cf-semistratifiable $w\Delta$-space) clearly $a_i\notin G_{n_0}$ for each $i=1, 2, \cdots, m$ But then $F(n, U)\cap G_{n_0}\subset F(n, U)\cap G_k=\{a_1, a_2, \cdots, a_m\}$ implies $F(n, U)\cap G_{n_0}=\emptyset$ which is a contradiction. Hence we can choose $x_{n_k} \in G_{n_k} \cap F(n_k, U)$ such that $n_k > n_{k-1} \geqq n_0$. Thus we have defined a subsequence $\{x_{n_k}: k=1, 2, \cdots\}$ of S which is Cauchy. For let $\varepsilon(>0)$ be given, then there an integer n_0 such that diam $G_{n_0} < \varepsilon$. For $i, j \geqq n_0$ we then have $x_{n_i} \in G_i \subset G_{n_0}$ and $x_{n_j} \in G_j \subset G_{n_0}$. Thus $d(x_{n_i}, x_{n_j}) \leq$ diam $G_{n_0} < \varepsilon$.

In light of the characterization of spaces with a G_δ-diagonal by Ceder [1] and Borges' study of spaces with a $\overline{G}_\delta$-diagonal(see [26]). Hodel [20] introduced the following definition.

Definition 5.8[20]. A space X has a G^*_δ-diagonal if there is a sequence $\mathscr{G}_1, \mathscr{G}_2, \cdots$ of open covers of X such that, for any two distinct points x and y of X, there is a n in N such that $y \notin st(x, \mathscr{G}_n)$. Such a sequence of open covers is called a G^*_δ-sequence for X.

In [27] Kullman proved that every regular θ-refinable space with a G_δ-diagonal has a $\overline{G}_\delta$-diagonal. Since every space with a $\overline{G}_\delta$-diagonal has a G^*_δ-diagonal, in [26] Hodel showed the following Lemma.

Lemma 5.9. Every regular θ-refinable space with a G_δ-diagonal has a G^*_δ-diagonal. In particular every regular semiatratifiable space has a G^*_δ-diagonal.

The next result relates the cf-stratifiable property to the G^*_δ-diagonal.

Lemma 5.10. Every (regular) cf-semistratifiable space has a G^*_δ-diagonal.

Proof. Let F be a cf-semistratification for X, and let $\mathscr{G}_n = \{U \subset X: U$ open in $X\}$, $\mathscr{G}_n$ open covers of X for each $n \in N$. To show that $\{\mathscr{G}_n\}_{n=1}^{\infty}$ is a G^*_δ-sequence for X, let x and y be distinct points of X. There are two filter base $\mathscr{U} = \{A_\alpha: \alpha \in \mathscr{A}\}$ and $\mathscr{B} = \{B_\beta: \beta \in \mathscr{B}\}$ converging to x and y respectively. Now we choose n in N such that if every $U \ni \mathscr{G}_n$, containing x, then there is an A_α such that $x \in A_\alpha \subset st(x, \mathscr{G}_n) \subset \cup F(n, U)$. Similary we choose $k \in N$ such that if every $V \in \mathscr{G}_k$, containing y, then there is a B_β such that $y \in B_\beta \subset st(k, \mathscr{G}_k) \subset \cup F(k, V)$. It follows that $A_\alpha \cap st(y, \mathscr{G}_k) = \emptyset$ and so $x \notin st(y, \mathscr{G}_k)$.

On the other hand $B_\beta \cap st(x, \mathscr{G}_n) = \emptyset$ this $y \notin st(x, \mathscr{G}_n)$. Thus $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is a G^*_δ-sequence for X.

Theorem 5.11. A space X is a cf-semistratifiable $w\Delta$-space if and only if it is developable.

Proof. Necessity; It follows from the above lemma 5.10 and that every $w\Delta$-space with a G^*_δ-diagonal is developable in Hodel [20].

Sufficiency; Let $\mathscr{H}_1, \mathscr{H}_2, \cdots$ be a $w\Delta$-sequence for X, and let $\mathscr{U} = \{A_n: n \in N\}$ be a convergent filter base for X. For eachpositive integer n, let $\mathscr{G}_n = \{G: G = (\bigcap_{i=1}^{n} H_i) \cap (\bigcap_{i=1}^{n} A_i), H_i \in \mathscr{H}_i: A_i \in \mathscr{U}\}$. To show that $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is a $w\Delta$-sequence with a cf-semistratifiable

$w\Delta$-sequence for X. We can choose a neighborhood $U(x)$ of x such that $x \in st(x, \mathscr{G}_n) \subset U(x)$ since $\mathscr{G}_1, \mathscr{G}_2, \cdots$ is a development for X, and choose a sequence $\langle x_n \rangle$ such that for all n, $x_n \in st(x, \mathscr{G}_n)$. Then $x_n \in U(x)$. This implies that $\langle X_n \rangle$ converges to x since $\mathscr{G}_{n+1}$ is an open refinement of $\mathscr{G}_n$ for all $n \in N$. Hence there is $G_n \in \mathscr{G}_n$ such that $x_n \in G_n \subset st(x, \mathscr{G}_n) \subset U(x)$. Suppose the filter base $\mathscr{U} = \{A_\alpha : \alpha \in \mathscr{A}\}$ converging to x has a cluster point p such that $x \neq p$. Then clearly there is a positive integer k such that for a neighborhood V of p, $V(p) \cap st(x, \mathscr{G}_k) = \emptyset$. Now for $n \geqq k$, $A_\alpha \subset st(x, \mathscr{G}_n) \subseteq st(x, \mathscr{G}_k)$ for all $\alpha \geqq \beta$, α, $\beta \in \mathscr{A}$ and so $A_\alpha \not\subset V(p)$ for all $\alpha \geqq \beta$. This constracts the fact that P is a cluster point of $\mathscr{U}$. Thus $\{\mathscr{G}_n : n=1, 2, \cdots\}$ is a cf-semistratifiable $w\Delta$-space.

Corollary 5.12. The following are equivalent for a regular $w\Delta$-space X;

(a) X is a Moore space

(b) X is cf-semistratifiable

(c) X is θ-refinable and has a G_δ-diagonal

(d) X has a G^*_δ-diagonal.

Proof. The implication (a)⇒(b) is due to Creede [2]. (b)⇒(c) follows from result by Hodel [20] (c)⇒(d) follows Lemma 5.10. (d)⇒(a) follows from theorem 5.11 above.

Definition 5.13[19]. A space X is an α-space if there is a function g from $N \times X$ into the topology of X such that for each $x \in X$,

(a) $\bigcap_{n=1}^{\infty} g(n, X) = \{x\}$

(b) if $y \in g(n, x)$ then $g(n, y) \subseteq g(n, x)$.

Such a function is called an α-function for X.

Lemma 5.14. The following are equivalent for a space X

(a) X is cf-semistratifiable

(b) There is a function g from $N \times X$ into the topology of X such that (1) for each $x \in X$ and $n \in N$, $x \in g(n, x)$; (2) if $x \in g(n, x_n)$ for $n=1, 2, \cdots$ then x is a cluster point of the sequence $\langle x_n \rangle$.

Proof. It is due to Hodel [4].

Theorem 5.15. In a reguler $w\Delta$-space, the following are equivalent for a space X.

(a) X is cf-semistratifiable

(b) X is an α-space.

Proof. Necessity; Since if X is cf-semistratifiable then X is a Moore space by corollary 5.12 and then every Moore space is an α-space since every subparacompact space with a G_δ-diagonal is an $\alpha^\#$-space and every $\alpha^\#$-space is an α-space in [4]. Sufficiency; Let $\mathscr{G}_1$,

$\mathscr{G}_2, \cdots$ be a $w\Delta$-sequence for X and let g be an α-function for X. Assume that for $x \in X$ and $n \in N$, $g(n+1, x) \subseteq g(n, x)$. For x in X and $n \in N$, let $h(n, x) = g(n, x) \cap st(x, \mathscr{G}_n)$. To show that the function h satisfies (b) of the above lemma.

Cleary (1) of (b) satisfied. To check (2), let $x \in h(n, x_n)$ for $n=1, 2, \cdots$ Then for $n=1, 2, \cdots$ $x \in st(x_n, \mathscr{G}_n)$ and so $x_n \in st(x, \mathscr{G}_n)$. Thus the sequence has a cluster point y. Suppose $y \neq x$. Now $\{y\} = \bigcap_{n=1}^{\infty} g(n, y)$ and so there is a $k \in N$ such that $x \notin g(k, y)$. Since y is a cluster point of $\langle x_n \rangle$ there is a $m \geqq k$ such that $x_m \in g(k, y)$. Since g is an α-function for X, $x_m \in g(k, y)$ implies $g(k, x_m) \subseteq g(k, y)$.

But $x \in h(m, x_m) \subseteq g(m, x_m) \subseteq g(k, x_m)$ and so $x \in g(k, y)$ which is a contradiction. Thus $x = y$ and x is a cluster point of $\langle x_n \rangle$.

Definition 5. 16. (a) Let X be a space and let g be a function from $N \times X$ into the topology of X such that for all $x \in X$ and $n \in N$, $x \in g(n, x)$. The space X is q-space [23] if $x_n \in g(n, x)$ for $n=1, 2, \cdots$ then the sequence $\langle x_n \rangle$ has a cluster point and the space X is called 1^0-countable space [20] if $x_n \in g(n, x)$ for $n=1, 2, \cdots$ then x is a cluster point of the sequence $\langle x_n \rangle$. (b) A space X is called a β-space [9] if there is a function g from $N \times X$ into the topology of X such that (1) for all $x \in X$ and $n \in N$, $x \in g(n, x)$, (2) If $x \in g(n, x_n)$ for $n=1, 2, \cdots$ then the sequence $\langle x_n \rangle$ has a cluster point. Such a function is called a β-function for X.

Theorem 5. 17. A $w\Delta$-space is a β-space and a cf-semistractifiable space is a β-space.

Proof. Straightforwad.

We can replace theorem 5. 2 of Hodel [20] by the following results whose proof can be omitted.

Theorem 5. 18. The following are equivalent for a regular space$X$;

(a) X is cf-semistratifiable

(b) X is a β-space with a G^*_δ-diagonal

(c) X is an α-space and a β-space.

Theorem 5. 19. A regular space is cf-semistratifiable if and only if it is a semistratratifiable β-space.

Proof. The necessity is clear. To show the sufficiency, let X be a regular semistratifiable β-space with a cf-semistratification F such that $Cl\ F(n+1, g(n+1, x)) \subset F(n, g(n, x))$ for all n and such that if $x \in F(n, g(n, y_n))$ for all $n \in N$ then the filter base $\mathscr{U} = \{g(n, y_n) : n \in N\}$ has a cluster point. Let y, $y_n \in X$ such that $y \in g(n, y_n) \in \mathscr{U}$ for $n \in N$. We wish to show that $\{g(n, y_n) : n \in N\}$ converges to y. The function g also has a characterization of semistratifiable spaces due to Theorem 1. 1 in Creede [2]. The filter base $\mathscr{U}$

has at least one cluster point, moreover every subsequence of $\mathscr{U}$ also has at least one cluster point.

Now let x be another cluster point of $\mathscr{U}$ distinct from y. Choose a subsequence of sets, $\{g(n_i, y_{n_i}): n_i \in N\} \subset \{g(n, y_n): n \in N\} = \mathscr{U}_i$ with $g(i, x)$ containing y_{n_i} for $i=1, 2, \cdots$ and $y_{n_i} \neq y$ for all i.

Since $Cl\ F(i+1, g(i+1, x)) \subset F(i, g(i, x))$, x is the only one cluster point of $\{g(n_i, y_{n_i}): n_i \in N\}$ it follows that $\mathscr{U}_i \to x$ so that there exists m such that $x \in g(n, \{x\} \cup \langle y_{n_i} \rangle)$ if $n > m$.

Take $k > m$. Then $y \notin g(m, y_k) \supset g(k, y_k)$, which is a contradiction. It follows that x is the unique cluster point of $\{g(n, y_n): n \in N\}$ since every subsequence of $\mathscr{U}_i$ has a cluster point, $\mathscr{U}$ converges to y.

Corollary 5.20. Let X be a cf-semistratifiable space. If for $U \subset X$, $F(n, U)$ (where F is a cf-semistratification for X) is a 1^0-countable subspace of X, then X is 1^0-countable.

Proof. Let $x \in X$, F a cf-semistratification for X and U open in X such that $x \in F(n, U)$. Since, for each subset U of X, $F(n, U)$ is a 1^0-countable subspace and X is T_1, and it is a cf-semistratifiable space, a countable collection which is a subfilter base $\mathscr{U}_n = \{A_n: n \in N\}$ of $\mathscr{U} = \{A_\alpha: \alpha \in \mathscr{A}\}$ converging to x in X may be found such that $Cl(A_{n+1}) \subset A_n \subset F(n, U)$ for each $n \in N$. And now let V be any open set containing x, then there is a natural number n such that $x \in A_n \cap F(n, U) \subset V \cap F(n, U)$. It follows that $Cl(A_{n+1} \cap F(n, U)) \subset Cl(A_{n+1}) \cap F(n, U) \subset A_n \cap F(n, U) \subset V \cap F(n, U)$. Note that $F(n, U) \cap A_n - V) = \emptyset$. Thus, $F(n, U) \cap (Cl(A_{n+1}) - V) = \emptyset$. We can put $\{G_m: m \in N\}$ to be a first-countable(modk) base for X. Since each subset is 1^0-countable subbase of X, there is a $m \in N$ such that $x \in G_m \subset X - (Cl(A_{n+1}) - V)$. It follows that $x \in A_{n+1} \cap G_m \subset V$ and that $\{A_n \cap G_m: n \in N, m \in N\}$ is a local base at x.

6. Metrization

In this section, we wish to give necessary and sufficient conditions for a cf-semistratifiable space to be metrizable.

Definition 6.1[9]. A system $G = \{g(n, x): x \in X, n \in N\}$ is called a graded system of open covers if

(a) $x \in g(n, x)$ and $g(n, x)$ is open for each $x \in X$ and each naturalnumber $n \in N$

(b) $g(n+1, x) \subseteq g(n, x)$ for all $n \in N$ and each $x \in X$, and

(c) $\{x\} = \cap \{g(n, x): n \in N\}$ for each $x \in X$.

A graded system of open covers $\{g(n, x): n \in N, x \in N\}$ is called a c-semistratification for X provide that $A = \cap \{g(n, A): n \in N\}$ for each closed compact set A where $g(n, A)$

$=\cup\{g(n, x): x\in A\}$. A space is c-semistratifiable if it has a c-semistratification.

Definition 6.2[13]. A space X is a wM-space if it has a sequence $\Delta=\{\mathscr{G}_n: n\in N\}$ of open covers of X such that if $x\in st^2(x, \mathscr{G}_n)$ for each n, the sequence $\langle x_n\rangle$ has a cluster point.

Definition 6.3[11]. A space X is said to be developable (mod k) if there exists a compact covering $\mathscr{K}$ of X and a sequence $\Delta=\{\mathscr{G}_n: n=1, 2, \cdots\}$ of open covers of X such that for each $x\in K\in\mathscr{K}$, $K\subset U$ where U is open, then there is a $n\in N$ such that $st(x, \mathscr{G}_n)\subset U$. A regular developable (mod k) space is called a Moore (mod k) space and Δ is called a development (mod k) for X.

From the above definition we can easily see that every wM-space is a $w\Delta$-space [5.4] and we can give the following theorem.

Theorem 6.4. A Freche't cf-semistratifiable space is c-semistratifiable.

Proof. Suppose that A is a compact subset such that $\cap\{g(n, A): n\in N\}\neq A$. Then there exists an x such that $x\in\bigcap_n\{g(n, A): n\in N\}-A$. We can choose A_α belonging to filter base $\mathscr{U}=\{A_\alpha: \alpha\in\mathscr{A}\}$ converging to $x\in X$ such that $x\in A_\alpha\subset g(n, A_\alpha)$. Let y be a cluster point of $\langle x_n\rangle$ in A_α, and the Freche'tness of the space guarantess the existence of a subsequence $\langle x_{n_i}\rangle$ of $\langle x_n\rangle$ in A_α which converges to y. That is, $\bigcap_{k\in N} g(k, A_\alpha)=\{y\}\cup\langle x_{n_i}\rangle\subset A$ implies $x\in A$. This is a contradiction.

Theorem 6.5. X is a Hausdorff cf-semistratifiable space if and only if it is a cf-semistratifiable space.

Proof. Necessity: Let x, y be two distinct points of X. There are open sets $g(n, x_n)$ and $g(n, y_n)$ of a graded system such that $x\in g(n, x_n)$, $y\in g(n, y_n)$ and $g(n,x_n)\cap g(n, y_n)=\emptyset$ and there are closed compact sets $\{y\}\cup\langle y_n\rangle$ and $\{x\}\cup\langle x_n\rangle$ for each x, $y(x\neq y)$. These satisfy Definition 1.1. Since a convergent filterbase are replaced with $\mathscr{U}=\{g(n, x_n): n\in N$, each $x\in g(n, x_n)\}$ and U_n is replaced with $g(n, \{x\}\cup\langle x_n\rangle)$.

Sufficiency: straightforward.

Theorem 6.6. A developable (mod k) space is a $w\Delta$-space.

Proof. Let $(X, \mathscr{K}, \mathscr{G})$ be a developable (mod k) space. We may assume that each $\mathscr{G}_{n+1}$ is a refiniment $\mathscr{G}_n$. Let $x_n\in st(x, \mathscr{G}_n)$ for each n. Assume $\langle x_n\rangle$ has no cluster point. Let $K\in\mathscr{K}$ containing x. Then it is shown that $\langle x_n\rangle\cap K$ is a finite set so that we may assune $x_n\notin K$ for all n. Since $X-\langle x_n\rangle$ is an open set containing K, there exists a positive integer k such that $st(x, \mathscr{G}_k)\subset X-\langle x_n\rangle$. This implies that $x_k\in st(x, \mathscr{G}_n)$, which is a contradiction.

Corollary 6.7. A regular space X is a Moore space if and only if X is a Fréchet cf-semistratifiable space and a Moore (mod k) space.

Proof. Straightforward.

Corollary 6.8. A regular space is metrizable if and only if it is a cf-semistratifiable wM-space.

Proof. Note that wM-space is a β-space and apply theorem 5.19 and Corollary 5, in [9].

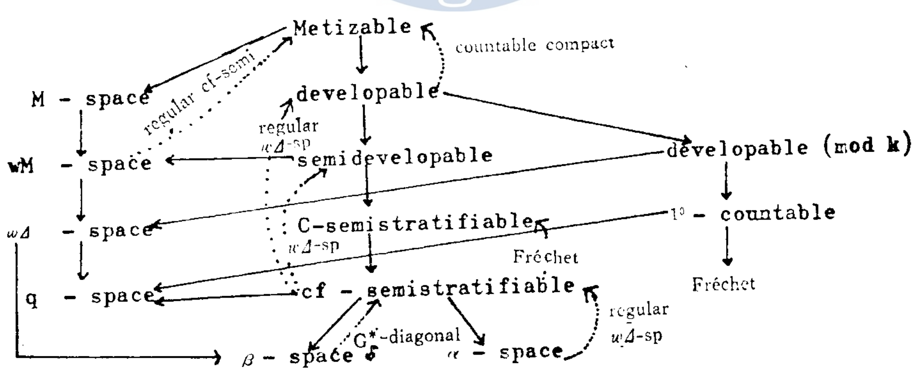
Theorem 6.9. In $w\Delta$-space, every countable compact cf-semistratifiable space is metrizable.

Proof. Since, in the $w\Delta$-space, a cf-semistratifiable space has a G^*_δ-diagonal and then it succeeds a Moore space and every countably compact Moore space is metrizable. Now apply that a cf-semistatifiable space is a Moore space in a regular space.

Example. The ordinal space $[0, \Omega]$ is a compact space and so it is a developable (mod k) but not metrizable. Since Ω belongs to $[0, \Omega]$, even if we would take sup $\{\alpha_n: \alpha_n < \Omega\}$, it is a member of a countable set but Ω is a uncountalbe set. Thus $\langle \alpha_n \rangle$ does not converge to Ω.

7. Summary

We can summarize the above results as follows. That is, the relationship between some of the classes of spaces considered in this paper can be summarized is a diagram as follows.



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