On Cs-Semidevelopable Spaces

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Cs-Semidevelopable空間에 관한 硏究

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Abstract

In this paper cs-semidevelopable spaces are defined and shown to be the same as the semimetrizable spaces. Strongly cs-semidevelopable space are defined in a natural way and proved to coincide with an important class of semi-metric space, namely those in which "Cauchy sequence suffice". These space are shown to possess as few other interesting properties. Probably the most significant of these are that a space X is a cf-semistratifisble $w\Delta$-space if and only of it is cs-semidevelopable and that the image of a cs-semidevelopable space under a closed continuous pseudo open is cs-semidevelopable.

要 約

1971年에 D.J. Lutzer가 semimetizable과 semistratifiable 空間을, 同年에 Charles C. Alexander는 semi-developable 空間 및 metric spaces의 Quotient像에 관한 硏究를 發表했다. 本論文에서는 semi-developable 空間에 eventually convergent sequence $\langle x_n \rangle$을 導入한 새로운 空間 cs-semidevelopable 空間을 紹介하여,

(1) cs-semidevelopable 空間이 semimetrizable 空間과 同値이다.

(2) cf-semistratifiable $w\Delta$-space가 cs-semidevelopable 空間이며,

(3) 閉連續 pseudo open 사상下의 cs-semidevelopable 空間의 像도 또한 cs-semidevelopable 空間임을 밝혔다.

1. Cs-semidevelopable spaces

Definition 1.1. (D_1). A development for a space X is a sequence

$\Delta = \{g_n | n \in N\}$

of open covers of X such that $\{st(x, g_n) | n \in N\}$ is a local base atx, for each $x \in X$. A space is developable if and only if there exists a development for the space,

Definition 1.2. Let $\Delta = \{g_n | n \in N\}$ be a sequence of (not necessarily open) convers of space X,

(D_2). Δ is a semidevelopment for X if and only if, for each $x \in X$, $\{st(x, g_n) | n \in N\}$ is a local system of neighborhoods at x.

(D_3). A semidevelopment of X is a strong-semidevelopment if and only if for each $M \subset X$ and $x \in \bar{M}$ there exists a descending sequence $\{G_n | n \in N\}$ such that $x \in G_n \in g_n$ and $G_n \cap M \neq \phi$.

(D_4) A semidevelopment Δ for X is a point-finite semidevelopment if and only if for each $x \in X$ and for each positive integer n, x is contained in only a finite number of sets in g_n.

(D_5) A semidevelopment Δ for X is a cs-semidevelopment if and only if for each convergent sequence $x_n \rightarrow x$ and for each open subset U containing $x \in X$, there is a positive integer k such that $x \in st(x, g_k) \subset U$ and $\langle x_n \rangle$ is eventually in $st(x, g_k)$.

A space is called semidevelopable if and only if ther exists a semidevelopment for X. Similarly, X is called strongly (and/or point finite) semidevelopable if and only if there exists a strong (and/or point-finite) semidevelopment for X

Finally, a space X is called cs-semidevelopable if and only if and only if there exists a cs-semidevelopment for X. Similarly that X is called strongly (and/or point-finite) cs-semidevelopable if and only if there exists a strong (and/or point-finite) cs-semidevelopment for X.

Proposition 1.3. In order that a sequence $\Delta = \{g_n | n \in N\}$ of cover of a space X be a cs-semidevelopment it is necessary and sufficient that for each $M \subset X$ and $x \in \bar{M}$ there exists a sequence $\{G_n | n \in N\}$ such that $x \in G_n \in g_n$ and $G_n \cap M \neq \phi$

Proof: Straightforward from Definition 1.2.

For late use, we note that every (point-finite and/or strongly)cs-semidevelopable space has a (point-finite and/or strong) cs-semidevelopment $\{g_n | n \in N\}$ having the property that

$g_{n+1} < g_n$ for each positive integer $n \in N$. Hence, whenever the existence of a cs-semidevelopment is assumed in a theorem. We may assume that it has the property mentioned above. cs-semidelopments having this property shall be called refining cs-semidevelopments.

Definition 1.4. A metric on a space X is a function d: $X \times X \to R$ (real numbers) satisfying the following conditions:

For each x, y, $z \in X$ and $\phi \neq M \subset X$

(1) $d(x, x) = 0$

(2) $d(x, y) > 0$ if $x \neq y$

(3) $d(x, y) = d(y, x)$

(4) $d(x, z) \leqq d(x, y) + d(y, z)$

(5) $x \in \bar{M}$ if and only if $d(x, M) = \inf \{d(x, m) | m \in M\} = 0$

Definition 1.5. A semi-metric on a space X is a function d: $X \times X \to R$ satisfying conditions (1), (2), (3) and (5) above By. a (semi-) metric space we mean a space X together with a specific (semi-) metric on X, In this paper, whenever the (semi-) metric is not specified it will be assumed to be denoted by the letter "d", the sphere about the point x of radius "ε" will be denoted by $S(x:\varepsilon)$. Note that spheres need not be open that $x \in \text{Int } S(x:\varepsilon)$ if $\varepsilon > 0$.

It should be noted that in most of our theorem the T_0 property is assumed. This is usually done to insure that a cs-semidevelopable space satisfies (2) in the previous definition which is satisfied in a semi-metric spaces.

Definition 1.6. Let (X, d) be a semi-metric space. A sequence $\{x_n | n \in N\}$ in X is a Cauchy sequence if and only if for each $\varepsilon > 0$ there exists an integer N_0 such that $d(x_n, x_m) < \varepsilon$ whenever $m, n > N_0$.

Note that because of the lack the triangle inequality not all convergent sequence in a semimetric space are necessarily Cauchy sequences.

2. Theorems for Cs-semidevelopable spaces

Theorem 2.1. A space X is semi-metrizable if and only if it is a cs-semidevelopable space.

Proof: Let $\Delta = \{g_n | n \in N\}$ be a refining cs-semidevelopment for the cs-semidevelopable space where, without loss of generality, $g_1 = \{X\}$. For $x, y \in X$, let $n(x, y)$ be the smallest integer n such that there is n_0 element of g_n containing both x and y. If no such integer exists let $n(x, y) = \infty$.

Define d: $X \times X \to R$ as follows. For $x, y \in X$, let $d(x, y) = 2^{-n(x,y)}$, where $2^{-\infty} = 0$. Then clearly, for every $x, y \in X$, $d(x,x)=0$ and $d(x,y)=d(y,x)$, Also if $x \neq y$, then, since X satisfies (D_5) in the previous Definition 1.2., there is an open set U containing one of the points, say x but not the other. Then there is an integer n such that $x \in st(x, g_n) \subset U$. Then $y \in U$ implies $y \in st(x, g_n)$ which implies $y \in st(x, g_i)$for each $i \geqq n$. It follows that $n(x,y) \leqq n$ and hence $d(x,y) \geqq 2^{-n} > 0$.

Now note that $S(x:2^{-n}) = st(x, g_n)$ for each $x \in X$ and each integer n. For $y \in S(x:2^{-n})$ if and only if $d(x,y) < 2^{-n}$ if and only if $n(x,y) > n$ if and only if there exists $G \in g_n$ such that $x, y \in G$ if and only if $y \in st(x, g_n)$. Now let $M \subset X$. Then $X \in \bar{M}$ if and only if $st(x, g_n) \cap M \neq \phi$ for each integer n if and only if $S(x:2^{-n}) \cap M \neq \phi$ for each integer n if and only if $d(x, M) = 0$ Hence, d is a semi-metric on X.

Conversely, assume that d is a semi-metric on X. For each positive integer n, let g_n be the collection of all sets of diameter less than $1/n$. Then for each n, $S(x: 1/n) = St(x, g_n)$. For let $y \in S(x:1/n)$. Then $G = \{x, y\} \in g_n$ implies $y \in st(x, g_n)$. On the other hand, let $y \in st(x, g_n)$. Then there is $G \in g_n$ such that $x, y \in G$, and therefore, $d(x,y) \leqq \text{diam } G < 1/n$ thus, $y \in S(x:1/n)$.

Now let U be an open set containing the point x. Then there is an integer n such that $x \in \text{Int } S(x:1/n) \subset S(x:1/n) \subset S(x_n: 1/n) \subset U$. Therefore, $x \in \text{Int } st(x, g_n) \subset st(x, g_n) \subset st(x_n, g_n) \subset U$ and $\langle x_n \rangle$ is eventually in $st(x, g_n)$. Hence $\{g_n | n \in N\}$ is a cs-semidevelopment for X.

Corollary 2.2. Every cs-semidevelopable space is T_1.

Proof: Since every cs-semidevelopable space implies T_0 semi-developable and moreover T_0 semidevelopable spaces succeed T_1-space.

Theorem 2.3. In a cs-semidevelopable space the following conditions are equivalent:

(1) For each $M \subset X$ and each $x \in \bar{M}$, there exists a descending sequence of sets $\{G_n | n \in N\}$ of arbitrarily small diameters such that for each n, $x \in G_n$ and $x \in G_n \cap U \neq \phi$.

(2) For each $M \subset X$and each $x \in \bar{M}$, ther exists a Cauchy sequence in M converging to x.

(3) Every convergent sequence has a Cauchy subsequence.

Proof: Let d be a semi-metric on X since every cs-semidevelopable space implies a semi-metric space.

(1) implies (3). Let $S = \{x_n | n \in N\}$ be a sequence in X converging to the point $x \in$X. It $x_n = x$ for infinitely many n, then clearly we can define a Cauchy subsequenc of S.

Otherwise let $M = \{x_n | n \in N\} \setminus \{x\}$. Then $x \in \bar{M}$ implies, by (1), that there is a descending sequence of sets $\{G_n | n \in N\}$ of arbitrarily small diameters such that for each n, $x \in G_n$ and $G_n \cap M \neq \phi$. We now define a subsequence of $\{x_n | n \in N\}$ inductively. Choose $x_{n_1} \in G_1 \cap M$. Suppose x_{n_i} has been chosen for each $i = 1, 2, \cdots, k-1$, such that $x_{n_i} \in G_i \cap M$

and $n_i > n_{i-1}$. Now observe that $G_k \cap M$ is infinite.

For suppose not: say $G_k \cap M = \{a_i, \cdots, a_m\}$. Then there exists $n_0 > K$ such that diam $G_{n_0} < \min \{d(x, a_i) \mid i = 1, 2, \cdots, m\}$. Clearly, $a_i \notin G_{n_0}$ for each $i = 1, 2, \cdots m$. But then

$$M \cap G_{n_0} \subset M \cap G_k = \{a_i, \cdots, a_m\} \text{ implies } M \cap G_{n_0} = \phi, \text{ which is a contradiction.}$$

Hence we can choose $x_{n_k} \in G_k \cap M$ such that $n_k > n_{k-1}$. Thus we have defined a subsequence $\{x_{n_k} \mid k \in N\}$ of S which is Cauchy. For let $\varepsilon > 0$ be given. Then there is an integer N_0 such that diam $G_{N_0} < \varepsilon$. For $i, j \geqq N_0$, we then have $x_{n_i} \in G_i \subset G_{N_0}$ and $x_{n_j} \in G_j \subset G_{N_0}$. Thus $d(x_{n_i}, x_{n_j}) \leqq$ diam $G_{N_0} < \varepsilon$.

(3) implies (2): Assume $M \subset X$ and $x \in \bar{M}$. Since X is first countable there is a sequence $\{x_n \mid n \in N\}$ in M which converges to x. By (3), this sequence has a Cauchy subsequence $\{x_{n_k} \mid k \in N\}$. Then $\{x_{n_k} \mid k \in N\}$ is a Cauchy sequence in M converging to x.

(2) implies (1): Let $M \subset X$ and assume $x \in \bar{M}$. Then, by (2), there is a Cauchy sequence $\{x_n \mid n \in N\}$ in M which converges to x. For each n, let $G_n = \{x_i \mid i \geqq n\} \cup \{x\}$. Then $\{G_n \mid n \in N\}$ is a descending sequence of sets of arbitrarily smally small diameters such that for each n, $x \in G_n$ and $G_n \cap M \neq \phi$.

Definition 2.4. A space X is strongly semi-metrizable if and only if a semi-metric satisfying any one of the conditions of the previous theorem can be realized on X.

Such a semi-metric is called a strong semi-metric.

Theorem 2.5. A space X is strongly semi-metrizable if and only if it is a strongly cs-semidevelopable space.

Proof: Let d be a strong semi-metric for X then, by Theorem 2,3 d satisfies condition (1). Now consider the cs-semidevelopment defined in Theorem 2.1. By the definition of Δ_d and the fact that d satisfies the condition (1), it follows immediately that Δ_d is a strong cs-semidevelopment.

Conversely, let $\Delta = \{g_n \mid n \in N\}$ be a refining strong cs-semidevelopment for X. Let d_Δ be the semi-metric on X as defined in Theorem 2.1 Observe that with this semi-metric, diam $G \leqq 2^{-n}$ for each $G \in g_n$ and $n \in N$. Thus it follows the definition of a strong semi-developmetnt that d_Δ-satisfies condition (1) of the previous theorem and hence all of the conditions.

Definition 2.6. A space X as a $w\Delta$-space if and only if there is a sequence $\{g_n \mid n \in N\}$ of open cover of X such that, for each $x \in X$, if $x_n \in st(x, g_n)$ for $n \in N$ then the sequence $\langle x_n \rangle$ has a cluster point. Such a sequence of open covers is called a $w\Delta$-sequence for X.

Theorem 2.7. A space X is a cf-semistratifiable $w\Delta$-space if and only if it is cs-semidevelopable.

Proof: Let F be a cf-semistratification for a space X, and let $\Delta=\{g_n|n\in N\}$ is a $w\Delta$-sequence for the space X. We can take a $st(x, g_n)$ such that $st(x, g_n)\subset A_\alpha\subset F(k, U)$, where A_α is an element of any filterbase in X.

Since from definition of filterbase, g_{n+1} is a open refinement of g_n for all n. Thus $\{st(x, g_n)|n\in N\}$ is a bocal system of neighbor hood at x, therefore $\{g_n|n\in N\}$ is a semidevelopment for X and moreover, there is an convergent sequence $\langle x_n\rangle$ is the space X since X is a $w\Delta$-space, there is a positive $k\in N$ such that $x\in st(x, g_n)$ and $x_n\in st(x, g_n)\subset U$, for all $n\in N$. Hence the semidevelopable space implies a cs-semidevelopable space as desired.

Conversely, let $\{\mathscr{H}_n|n\in N\}$ be an open covers of X, and let $\mathscr{U}=\{A_\alpha|\alpha\in\mathscr{A}\}$ be a convergent filter base for X. For each positive integer n, let $g_n=\{G|G=(\bigcap_{i=1}^{n} H_i)\cap(\bigcap_{i=1}^{n} A_{\alpha i}), H_i\in\mathscr{H}_i, A_{\alpha i}\in\mathscr{U}\}$, then $\{g_n|n\in N\}$ is a cs-semidevedopment for X. To show that $\{g_n|n\in N\}$ is a $w\Delta$-sequence with a cf-semistratification for X. We can choose a neighborhood $U(x)$ of x such that $x\in st(x, g_n)\subset U(x)$. Since $\{g_n|n\in N\}$ is a semidevelopment for X, and choose a sequence $\langle x_n\rangle$ such that for all n, $x_n\in st(x, g_n)$, then $x_n\in U(x)$ this implies that $\langle x_n\rangle$ converges to x since g_{n+1} is an open refinement of g_n for all $n\in N$. Hence there is $A_n\in g_n$ such that $x_n\in A_n\subset st(x, g_n)$. Suppose the filter base $\mathscr{U}=\{A_\alpha|\alpha\in\mathscr{A}\}$ converging to x has a cluster point p such that $x\neq p$. Then clearly there is a positive integer k such that for a neighborhood V of p, $V(p)\cap st(x, g_k)=\phi$. Now for $n\geqq k$, $A_\alpha\subset st(x, g_n)\subseteq st(x, g_k)$ for all $\alpha\geqq\beta$, x, $\beta\in\mathscr{A}$ and so $A_\alpha\cap V(p)=\phi$ for all $\alpha\geqq\beta$. This constracts the fact that p is a cluster point of $\mathscr{U}$. Thus $\{g_n|n\in N\}$ is a cf-semistratifiable $w\Delta$-space.

Corollary 2.8. Let X be a reqular $w\Delta$-space. Then X is an α-space if and only if X is a cs-semidevelopable space.

3. Mappings

Charles C. Alexander introduced the concept of pseudo map.

Definition 3.1. Let X and Y be topological spaces, Then a surjective map from X onto Y is pseudo-open if and only if for each $y\in Y$ and each open neighborhood U of $f^{-1}(y)$ in X, $y\in\text{Int } f(U)$.

Theorem 3.2. The image of a cs-semidevelpable space a under continuous pseudo-open map is cs-semidevelopable.

Proof: Let f be a continuous pseudo-open map from a cs-semidevelopable space X onto a space Y and $\Delta=\{g_n|n\in N\}$ a cs-semidevelopment for X.

For each open V_n containing a point y of Y and for all n, we can put $f^{-1}(V_n)=st(x, g_n)$.

Since Δ is a cs-development for X and f is continuos, let U be any open set in X including $f^{-1}(V_n)$, then there is an convergent sequence $\langle x_n \rangle$ converging a point x belonging to $f^{-1}(y)$ in U, where $\langle y_n \rangle$ converges to y in Y. On the other hand, by Definition 1.2. there exists a $n_0 \in N$ such that $st(x, g_n)$ is contained in for all $n > n_0$ and $\langle x_n \rangle$ is eventually in $st(x, g_{n_0})$. that is, $y \in f(st(x, g_n)) \subset \mathrm{Int}\, f(st(x, g_{n_0}))$ and therefore g_n is contained Int $f(st(x, g_{n_0}))$ for all $n > n_0$.

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