A Study on Polytopes and Triangulated Spaces

Sung Ryong Yoo

1. Preliminary

In the present paper, we would study the topological structure for the polytope which has become a generic term used to denote those subsets of a Euclidean space, such as polygons or polyhedra which are constructed with rectilinear elements.

Above all things, we introduce that some examples have been spaces which are homeomorphic to some polytope and refer here to such things as geometric complexes, star topologies, barycentric subdivision and the basic geometry of polytope. And then we are
going to study simplicial mappings

2. Geometric complex and polytopes

Definition 2.1. A set \( \{a_0, a_1, \ldots, a_s\} \) of vector in \( E^s \) is pointwise independent provided that the vectors \( a_1-a_0, a_2-a_0, \ldots, a_s-a_0 \) are linearly independent. Here, \( \{v_1, v_2, \ldots, v_k\} \) (subset vector space, \( k \) finite) is linearly independent iff \( \sum_{i=1}^{k} f_i v_i = 0 \), \( f_i \in \text{Field} \rightarrow f_i = 0 \) \( (i=1, \ldots, k) \)

Definition 2.2. Let \( A = \{a_0, a_1, \ldots, a_s\} \) be a set of \( k+1 \) pointwise independent point in \( E^s \). And we denote by \( \{h \in H^s | h = \sum_{i=0}^{s} f_i a_i, \sum_{i=0}^{s} f_i = 1, f_i \geq 0, a_i \in A \} \).

Then we define \( S_k \) by geometric \( k \)-simplex in \( E^s \) determined by \( A \).

(Example 2.1‘) (1) \( S_0 \) : a set of only one point, \( \{a_0\} \)
(2) \( S_1 \) : a closed line segment, \( a_0 \ldots a_1 \).
(3) \( S_2 \) : a closed triangular plane region.
(4) \( S_3 \) : a closed solid tetrahedron, so on.

Definition 2.3. \( H^s(\subset E^s) \) is \( k \)-dimensional hyperplane iff there exists \( a_0, \{a_1, a_2, \ldots, a_s\} \) of \( E^s \) such that \( H^s = \{h | h = a_0 + \sum_{i=1}^{s} t_i a_i, t_i \in E^1 \} \). As it were, if \( a_0 = 0 = (0, 0, \ldots, 0) \), then \( H^s \) is \( k \)-dimensional vector subspace of \( E^s \).

And so \( H^1 \subset E^2 \), then \( H^1 = \{(x, 0) | x \in E^1 \} \) is one dimensional hyperplane for \( E^2 \).

We denote open geometric \( k \)-simplex in \( E^s \) (determined by \( A \)) by \( S_k = \{h \in H^s | h = \sum_{i=0}^{s} f_i a_i, \sum_{i=0}^{s} f_i = 1, f_i > 0, a_i \in A \} \).

Lemma 2.4. Let \( A = \{a_0, a_1, \ldots, a_k\} \) \( (k \leq n, A \subset E^s) \) be pointwise independent. Then there exists unique \( k \)-dimensional hyperplane such that

(1) \( A \subset H^k \)

(2) \( h \in H^k \rightarrow h = a_0 + \sum_{i=1}^{k} g_i (a_i - a_0) \) \( (h \neq 0 \Rightarrow g_i : \text{unique}) \)

Proof: (1) Let \( H^k \) be \( \{h | h = a_0 + \sum_{i=1}^{k} g_i (a_i - a_0), g_i \in E^1 \} \).

Then (i) \( H^k \) is \( k \)-dimensional hyperplane by Definition 2.3.

(ii) \( \forall \alpha_j \in A, a_j = a_0 + \sum_{i=1}^{k} \delta_{ij} (a_i - a_0) \) where \( \delta_{ij} \) is Kronecker delta. Hence \( a_j \) belongs to \( H^k \) for all \( j \).

Thus \( A \subset H^k \).
(2) \( h-a_0=\sum_{i=1}^{h} g_i(a_i-a_0) \), \( h \in H' \) is \( k \)-dimensional subspace of \( E^s \) with basis \( \{a_i-a_0\} \). Hence \( g_i \) is unique for \( h \neq 0 \). And then we must claim the uniqueness of \( H' \).

Suppose that we have the other \( k \)-dimensional hyperplane \( F' \) such that

1. \( A \subseteq F' \)

2. \( \forall p \in F' \rightarrow p = a_0 + \sum_{i=1}^{h} g_i(a_i-a_0) \) (If \( p \neq 0 \Rightarrow g_i \) : unique). Then there exist \( B = \{b_1, b_2, \ldots, b_s\} \) and \( b_s \) such that \( \{b_1, b_2, \ldots, b_s\} \) is linearly independent and \( F' = \{p \mid p = b_0 + \sum_{i=1}^{h} f_i b_i\} \).

Since \( A \subseteq F' \), there exists \( f_i(\in E^s) \) such that \( a_i = b_0 + \sum_{i=1}^{h} f_i b_i, (j = 0, 1, 2, \ldots, k) \) for every \( a_i \).

Putting \( a_0 = b_0 + \sum_{i=1}^{h} f_0 b_i \), we gain \( a_i = a_0 + \sum_{i=1}^{h} (f_i - f_0) \cdot b_i \).

Since both \( B = \{b_1, b_2, \ldots, b_s\} \) and \( \{a_1-a_0, a_2-a_0, \ldots, a_k-a_0\} \) are linearly independent, there exists unique \( b_i = \sum_{j=1}^{h} g_i \times (a_j-a_0) (i = 1, 2, \ldots, k) \) and so,

\[ p = b_0 + \sum_{i=1}^{h} f_i \left( \sum_{j=1}^{h} g_i (a_j-a_0) \right) \text{ belongs to } H' \]

Hence \( F' \subset H' \), similarly \( H' \subset F' \). Thus \( F' = H' \).

Lemma 2.5. Let \( \langle p_0 p_1 \cdots p_s \rangle \) be the geometric \( k \)-simplex determined by a set \( A = \{p_0, p_1, \cdots, p_s\} \) of \( k+1 \) pointwise independent points of \( E^s \).

Then \( \langle p_0 p_1 \cdots p_s \rangle \) is the convex hull of the set \( A \).

Proof: To show that \( \langle p_0 p_1 \cdots p_s \rangle \) is convex, let \( \sum_{i=0}^{s} x_i p_i \) and \( \sum_{i=0}^{s} y_i p_i \) belong to \( \langle p_0 p_1 \cdots p_s \rangle \) where \( \sum_{i=0}^{s} x_i = 1 \) and \( \sum_{i=0}^{s} y_i = 1 \). We would show that \( \sum_{i=0}^{s} t x_i p_i + \sum_{i=0}^{s} (1-t) y_i p_i = \sum_{i=0}^{s} ((t x_i + (1-t) y_i) p_i \text{ belongs to } \langle p_0 \cdots p_s \rangle \). But \( \sum_{i=0}^{s} ((t x_i + (1-t) y_i) p_i = t \sum_{i=0}^{s} x_i + (1-t) \sum_{i=0}^{s} y_i = t + (1-t) = 1 \).

Hence \( \sum_{i=0}^{s} (t x_i + (1-t) y_i) p_i \text{ belongs to } \langle p_0 \cdots p_s \rangle \).

And then we verify that \( \langle p_0 p_1 \cdots p_s \rangle \) is convex hull. Now, let \( B \) be a convex set containing \( A \) and \( \sum_{i=0}^{s} x_i p_i \) belong to \( \langle p_0 p_1 \cdots p_s \rangle \).

To show \( \sum_{i=0}^{s} x_i p_i \in B \left( \sum_{i=0}^{s} x_i = 1 \right) \), by induction,
1'st-step; Suppose \( k=0 \Rightarrow \langle p_0 \rangle = \{ p_0 \} = A \subset B \). Hence \( p_0 \in B \).

2nd-step; Suppose that it is true for \( k-1 \) the same as the above assumption.

3rd-step; Let \( x_i = \sum_{i=0}^{k} x_i p_i \) be a point in \( \langle p_0 \cdots p_k \rangle \).

Then if \( x_i = 0 \Rightarrow \sum_{i=0}^{k} x_i p_i = \sum_{i=0}^{k-1} x_i p_i + x_k p_k = \sum_{i=0}^{k-1} x_i p_i \), that is, \( \sum_{i=0}^{k} x_i p_i \in B \) by 2nd-step. And then

\[
\text{if } x_i \neq 0, \quad \sum_{i=0}^{k} x_i p_i = \sum_{i=0}^{k-1} x_i p_i + x_k p_k = (1-x_k) \sum_{i=0}^{k-1} x_i p_i + x_k p_k
\]

\[
= (1-x_k) \sum_{i=0}^{k-1} \frac{x_i}{1-x_k} p_i + x_k p_k = (1-x_k) p_i + x_k p_k.
\]

But then, \( \frac{1}{1-x_k} \sum_{i=0}^{k-1} x_i = \frac{x_0 + \cdots + x_k}{1-x_k} = \frac{1}{1-x_k} = 1 \), and since

\[\sum_{i=0}^{k} x_i p_i \in B \text{ belongs to } B. \text{ Hence } \sum_{i=0}^{k} x_i p_i \text{ belongs to } B. \text{ Thus } B \text{ is the convex hull.}\]

So to speak, we obtain that for a given subset \( A \) of \( E^n \), the convex hull of \( A \) is the intersection of all convex subsets containing \( A \). And so the convex hull of any subset \( A \) of \( E^n \) is convex.

From now on, we introduce some notations and examples. We represent that two geometric simplexes, \( S^n \) and \( s^m \), \( m \leq n \), are properly joined, if either

1. \( s^n \cap s^n = \emptyset \) or
2. \( s^n \cap s^n = s^k(0 \leq k \leq n) \)

where \( s^k \) is a subsimplex of both \( s^n \) and \( s^m \). And we denote a geometric \( n \)-simplex \( s^n \) by

\[\langle p_0 p_1 \cdots p_n \rangle \]

we call \( \langle p_0 \cdots p_k \rangle \) to be \( k \)-face of \( s^n \) and then geometric complex \( K \) is expressed by a (countable) collection of faces of geometric simplexes properly joined such that for all \( s^n \in K \), every \( s^n \) is a face of \( s^n \) and belongs to \( K \).

**Example 2.2.** \( \langle p_0 \cdots p_3 \rangle \) is a face of \( \langle p_0 \cdots p_5 \rangle \), which is obtained by deleting only \( p_4 \). As it were, in simplex \( s^3 = \langle p_0 p_1 p_3 \rangle \), we have \( \langle p_0 p_1 p_4 \rangle = \langle p_0 p_4 \rangle \) which is a face of \( \langle p_0 p_1 p_4 \rangle \) as the following figure 1.

![Figure 1](image1.png)

![Figure 2](image2.png)

**Example 2.3.** Let \( s^2 \) be \( \langle p_0 p_1 p_3 \rangle \). Then \( K = \langle \langle p_0 \rangle, \langle p_1 \rangle, \langle p_3 \rangle, \langle p_0 p_1 \rangle, \langle p_1 p_3 \rangle \rangle \).
\( \langle p_0 p_3 \rangle, \langle p_0 p_1 p_2 \rangle \) is the geometric complex. Suppose that \( K = \{ \langle p_0 \rangle, \langle p_1 \rangle, \langle p_0 p_1 \rangle, \langle p_1 p_2 \rangle, \langle p_2 p_0 \rangle, \langle p_0 p_1 p_2 \rangle, \langle p_3 \rangle, \langle p_3 p_0 \rangle \} \) as the above figure 2, then \( K \) is the collection of simplexes but not geometric complex since \( \langle p_1 p_3 \rangle \cap \langle p_3 p_0 \rangle = \emptyset \) is not a face of any simplex properly joined.

Definition 2.4. (1) \( K \) is defined the topological geometric complex by a (countable) collection of properly joined topological geometric simplex such that for all \( s \in K \), every \( s' \) is a face of \( s \) and belongs to \( K \). And we define a topology for a particular class of complex.

(2) we define \( St(\sigma) = \{ \sigma \in K | \sigma : \text{a face of simplex } s \} \) by the star of a simplex \( s \) and so we denote \( K(\text{star-finite complex}) \) by \( \forall \sigma \in K, St(\sigma) \) is finite.

(3) \( \mathcal{T} \) is a star topology of a star-finite complex \( K \) if and only if \( \mathcal{T} \) has a base \( \mathcal{B} \) such that.

\[ \mathcal{B} = \{ X | X \cap \sigma, \ i \ \text{finite}, \ \sigma : \text{simplexes } X \cap \sigma \text{ is open in } \sigma, \forall \sigma \in K \} \]

Remark: \( |K| = \bigcup St(\sigma) = \bigcup \{ \sigma | \sigma \in K, K : \text{star finite complex} \} \) called the geometric carrier of the complex \( K \). And let \( (\bigcup \{ \sigma | \sigma \in K \}, \mathcal{T}) \) be a topological space and if the open star of a simplex \( \sigma (\in K) \) is denote by \( St(\sigma) \) such that \( St(\sigma) = \text{the interior of carrier of } St(\sigma), \text{that is, the collection of the open subset of } |K| \).

Example 2.4. Let \( K = \{ \langle p_0 \rangle, \langle p_1 \rangle, \langle p_2 \rangle, \langle p_0 p_1 \rangle, \langle p_1 p_2 \rangle, \langle p_2 p_0 \rangle, \langle p_0 p_1 p_2 \rangle \} \) be star finite complex of \( S^2 \).

Then \( St(p_0) = \{ \langle p_0 \rangle, \langle p_0 p_1 \rangle, \langle p_0 p_2 \rangle, \langle p_0 p_1 p_2 \rangle \} \) and each element of \( St(p_0) \) is carrier and we suppose that topology for each element, we gain topological space \( (\langle p_0 \rangle, \mathcal{T} \langle p_0 \rangle) \) which is carrier.

Example 2.5. In the adjacent figure 3, \( St(s_1) = K_1 - \{ \cup s \} \)

but

\[ St(s_1) = \{ \langle p_0 p_2 \rangle, \langle p_1 p_2 \rangle, \langle p_2 p_0 \rangle \} \]

But we could obtain \( St(\sigma) = St(\sigma) \) as the following example,

Example 2.6. In the adjacent figure 4, (that is, in topological space \( (|K|, \mathcal{T}_K) \),

\[ St(\langle p_0 \rangle) \text{ is the interior of the carrier of } St(\langle p_0 \rangle) \]

\[ = \{ \langle p_0 p_1 p_2 \rangle, \langle p_1 p_2 \rangle \}, \text{i.e.} \]

\[ \text{fig. 3} \]

\[ \text{fig. 4} \]

3. Barycentric Subdivision

Now we introduce a standard technique used for
producing a triangulation of a given polytope such that the new triangulation is finer than the original. This subdivision is presented first for a complex consisting of a single simplex \( s^* = \langle p_0, p_1, \ldots, p_n \rangle \) together with all of its faces. Such a complex is called the closure of a simplex and denote by \( \text{Cl}(s^*) \).

**Definition 3.1.** \( \text{cl}(s^*) = \{ s \mid s : \text{face of } s^* = \langle p_0, p_1, \ldots, p_n \rangle \} \) is called the closure of a simplex \( s^* \). And \( A = \{ p_0, p_1, \ldots, p_n \} \) are assumed to be pointwise independent and that

\[
S^*_A = \{ h \mid h = \sum_{i=0}^n x_i p_i, \sum_{i=0}^n x_i = 1, x_i \geq 0 \} = \langle p_0 p_1, \ldots, p_n \rangle
\]

is a geometric \( n \)-simplex and \( S^*_A \subseteq E^n \).

The collection of all points \( \hat{s}^*_{j} \), \( k = 0, 1, 2, \ldots, n, \) where \( \alpha_k \) is the number of \( K \)-simplices in \( \text{Cl}(s^*) \), will be the vertices of a new complex \( K' \), the first barycentric subdivision of \( K = \text{Cl}(s^*) \). Now returning to the vertices \( \hat{s}^*_{j} \), we will take a subset of this points to be vertices of a simplex in \( K' \), \( \langle \hat{s}^*_{1}, \ldots, \hat{s}^*_{n} \rangle \), iff \( s_1 < s_2 < \cdots < s_n \) in \( K \).

**Lemma 3.2.** If \( \{ \hat{s}^*_{j} \mid k = 0, 1, 2, \ldots, n, j = 1, 2, \ldots, \alpha_k \} \), where \( \alpha_k \) is the number of \( k \)-simpex in \( \text{Cl}(s^*) \), is the vertices of a new complex \( K' \), then the number of face for \( K' \) is \( 2^{n+1} - 1 \).

**Proof:** \( \hat{s}^*_{0}, \hat{s}^*_{1}, \ldots, \hat{s}^*_{n} \); \( s \); \( s \in C_k \)

\[
\hat{s}^*_{0}, \hat{s}^*_{1}, \ldots, \hat{s}^*_{n}
\]

Thus \( \sum_{k=1}^{n+1} C_k + \cdots + C_{n+1} = 2^{n+1} - 1 \)

**Remark:** The point \( \hat{s}^* \) is called the barycenter of the simplex \( s^* \) and in the centroid of the vertices \( p_i \) with equal weights assigned to each, \( i.e., \), let \( \hat{s}^* = \langle p_0, p_1, \ldots, p_n \rangle \) be a face \( s^* = \langle p_0 p_1, \ldots, p_n \rangle \), then \( \hat{s}^* = \frac{1}{\sum_{i=0}^{n} x_i} \) weight \( (j = 1, 2, \ldots, k) \).

\( \langle \text{Example 3.1} \rangle \) Let \( s^* \) be \( \langle p_0 p_1 p_2 \rangle \). Then \( \hat{s}^* = \frac{1}{3} \sum_{i=0}^{2} \)

\[
x_i p_i, \sum_{i=0}^{2} x_i = 1, \text{ and } x_0 = x_1 = x_2 \text{ like as the adjacent figure 5.}
\]

And so,

\[
K = \{ s_0^*, s_1^*, s_2^*, s_3^*, s_4^*, s_5^* \} = \{ \langle p_0 \rangle, \langle p_0 p_1 \rangle, \langle p_1 \rangle, \langle p_0 p_2 \rangle, \langle p_1 p_2 \rangle, \langle p_0 p_1 p_2 \rangle \}
\]

\[
K' = \{ \langle \hat{s}_0^* \hat{s}_1^* \hat{s}_2^* \rangle \langle \hat{s}_0^* \hat{s}_3^* \hat{s}_4^* \rangle \langle \hat{s}_5^* \rangle \}
\]

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where \( K^1 \) is the \( 1^n \) barycentric subdivision of \( K \).

As in the above fig. 5, this subdivision may now be done for each simplex of any geometric complex \( K \) and define a new complex \( K^1 \), the first barycentric subdivision of \( K \).

It is evident that the geometric carrier of \( K \) and \( K^1 \) are identical.

Now we define the mesh of a geometric complex \( K \) by \( \sup(d|d) \) the diameter of every simplex of \( K^1 \).

**Lemma 3.2.** Let \( s^i \) be a geometric \( k \)-simplex.

Then \( d(s^i) \) is equal to the length of its longest edge (or 1-face).

Proof: Suppose \( A = \{p_0, p_1, \ldots, p_n\} \), the set of all vertices of \( s^i \). Then \( s^i = \langle p_0, p_1, \ldots, p_n \rangle = \bigcap_{i=1}^{n} B_i \), where \( B_i \) contains \( A \), is convex hull of \( A \) by Lemma 2.5. \( d(s^i) = d(A) = d(a, b) \) where \( a, b \) are endpoint respectively of 1-face since \( d(A) = d\left( \bigcap_{i=1}^{n} B_i \right) \) (\( A \subset B_i, 1 \forall i \)).

Thus \( d(A) \) is equal to 1-face of the simplex.

**Theorem 3.3.** Let \( K \) be a geometric complex, finite dimension, \( n \) and finite mesh \( \lambda \) and let \( m \) be the mesh of its \( 1^n \) barycentric subdivision \( K^1 \).

Then \( m(K^1) \leq \frac{n}{n+1} \lambda \).

Proof: Let an arbitrary \( s^i \) belong to \( K \), \( s^i = \langle p_0, p_1, \ldots, p_n \rangle \). Then the barycenter \( \hat{s}^i \) of \( s^i \) equals to \( \frac{1}{k+1} \langle p_0 + \ldots + p_n \rangle \), where \( \frac{1}{k+1} \) is the barycentric coordinates of \( \hat{s}^i \) since \( \sum_{i=0}^{k} f_i = 1, f_0 + \ldots + f_k = 1 \) from \( f_0 = \ldots = f_k \) thus \( f_i = \frac{1}{k+1} \).

Now then, let \( \langle \hat{s}^i \rangle \) be an arbitrary 1-simplex of \( K^1 \) in the subdivision of \( s^i \) and \( s^i \) be a face of \( s^i \) in \( K \) as \( s^i \langle \hat{s}^i \rangle \).

Put \( \mu = \text{the length of } \langle \hat{s}^i \rangle \). If \( s^i \) is \( \langle p_0, \ldots, p_i \rangle \) then \( s^{i+1} = \langle p_i+1, \ldots, p_n \rangle \) and if \( \hat{s}^i \hat{s}^{i-1} \) is the line segment from \( \hat{s}^i \) to \( \hat{s}^{i-1} \), then \( \langle \hat{s}^i, \hat{s}^{i-1} \rangle \supsetneq \langle \hat{s}^i \rangle \).

And so,

the weight of the barycenter \( \hat{s}^i = \frac{1}{k+1} [p_0 + \ldots + p_i] = \frac{i+1}{k+1} \) while \( s^{i-1} \) has weight \( \frac{k-i}{k+1} \) as centroids of the vertices, since \( \hat{s}^{i-1} \) has \( \frac{(k-i+1)}{k+1} \) of the vertices and its weight is \( \frac{1}{k+1} \times (k-i) \).

In the result, \( \hat{s}^i = \frac{i+1}{k+1} \hat{s}^i + \frac{k-i}{k+1} \hat{s}^{i-1} \)

\( = \frac{1}{k+1} [p_0 + \ldots + p_i] \),

where \( \hat{s}^i \) is the centroid of these two particles as the below figure 6.
Putting the length of line segment from $s^i$ to $s^{i-1}$ by $l(\tilde{s}^i \tilde{s}^{i-1}) = \rho$, then we obtain

\[
\left( \frac{i+1}{k+1} \right) \mu = \frac{k-i}{k+1} (\rho - \mu)
\]

or

\[
\left( \frac{i+1}{k+1} + \frac{k-i}{k+1} \right) \mu = \frac{k-i}{k+1} \rho, \text{ i.e., } \mu = \frac{k-i}{k+1} \rho \]

since $\rho$ does not exceed the diameter of $s^i$, we conclude that

\[
\mu \leq \frac{k-i}{k+1} \leq \frac{k}{k+1} \lambda \leq \frac{n}{n+1} \lambda
\]

thus 

\[
m(K^i) \leq \frac{n}{n+1} \lambda
\]

Corollary 3.3. If $K^{(h)}$ is the $k^{th}$-barycentric subdivision of $K$, $K$ is $n-$dimensional geometric complex and the mesh of $K$ is $\lambda (< \infty)$, then $m(K^{(h)})$ converges to zero as $k \to \infty$.

4. Simplicial mappings

Definition 4.1. Let $|K|$ and $|L|$ be polytopes with triangulations $K$ and $L$ respectively.

Putting $K_r = \{p_i \mid p_i : \text{vertex of } K\}$ and $L_r = \{\sigma_i \mid \sigma_i : \text{vertex of } L\}$.

we define transformation $f$ from $K_r$ into $L_r$ ($f$: a possibly many-to-one), satisfying conditions that if $\langle p_0 \cdots p_s \rangle$ is a simplex of $K$, then $f(\langle p_0 \cdots p_s \rangle)$ (not necessarily distinct) are the vertices of a simplex of $L$ and that the function $f$ is a continuous extension if and only if $f(x) = f(\sum_{i=0}^{s} x_i p_i) = \sum_{i=0}^{s} x_i f(p_i)$ such that $x = \sum_{i=0}^{s} x_i p_i$,

\[
\sum_{i=0}^{s} x_i = 1, \; x_i \geq 0 \text{ if } s^* = \langle p_0 \cdots p_s \rangle \text{ is a simplex of } K \text{ and } x \in s^*.
\]

The above mapping $f$ is called a simplicial mapping.

Lemma 4.2. A simplicial mapping $f$ is continuous.

Proof: Let $x$ belong to $|K|$. Then there exists the $x$ in $s^* = \langle p_0 p_1 \cdots p_s \rangle$ which is belonged to $K$, and so,

\[
f(x + \Delta x) - f(x) = \sum_{i=0}^{s} (x_i + \Delta x_i) f(p_i) - \sum_{i=0}^{s} x_i f(p_i) = \sum_{i=0}^{s} \Delta x_i f(p_i).
\]

Hence $\sum_{i=0}^{s} \Delta x_i f(p_i)$ conver-
ges to zero when \( Ax_i \) approaches to zero.

From the above definition 4.1., we note that the mapping \( f \) from \( K_n \) into \( L_n \) such that \( \langle p_0, \ldots, p_n \rangle \in K, \ f(p_0), \ldots, f(p_n) \) are vertices of a simplex of \( L \) is a simplicial mapping.

**Lemma 4.3.** In a Euclidean space \( E^n \), let \( \{p_n\} \) and \( \{q_n\} \) be two sequences in polytope \( |K| (\subseteq E^n) \) such that \( \lim_{n \to \infty} p_n = p, \ \lim_{n \to \infty} q_n = q \) respectively, and let \( p_0q_n \) be the length of \( [p_n, q_n] \) which is the line segment from \( p_n \) to \( q_n \). If \( x_n \) belongs to \( [p_n, q_n] \) for each \( n \), and if there exists \( \lim_{n \to \infty} d(x_n, p_n) \) as \( n \to \infty \), then there exists an \( x \) on \( [p, q] \) such that

\[
(1) \lim_{n \to \infty} d(x_n, p_n) = d(x, p) \quad \text{and} \quad (2) \lim_{n \to \infty} x_n = x.
\]

Proof: Put \( p_n = (p_1, p_2, \ldots, p_n), q_n = (q_1, q_2, \ldots, q_n) \) and \( x_n = (x_1, x_2, \ldots, x_n), x = (x_1, x_2, \ldots, x) \).

Then \( d(x, p) = \sqrt{\sum_{i=1}^{n} (x_i - p_i)^2} \) and \( x_n = (1 - t_n)p_n + t_nq_n \) \((0 \leq t_n \leq 1)\)

Letting \( t_n = \frac{1}{2} \), we obtain \( x_n = \frac{1}{2} p + \frac{1}{2} q = x \). i.e., \( x \in [p, q] \), since \( x_n = \left( \frac{1}{2} \right) + \frac{1 - 1}{n+1} p_n \) \((0 \leq t_n \leq 1)\). While \( \lim_{n \to \infty} x_n = (x_1, x_2, \ldots, x) = x \) and \( \lim_{n \to \infty} p_n = \lim_{n \to \infty} (p_1, p_2, \ldots, p_n) = (p, p, \ldots, p) = p \)

The formula 1 becomes \( \sqrt{\sum_{i=1}^{n} (x_i - p_i)^2} = d(x, p) \).

To show the existence of \( \lim_{n \to \infty} d(x_n, p_n) \), we verify that there exists \( t_n = \frac{1}{2} - \frac{1}{n+1} \).

Now, assume \( (x_n - p_n) = t_n(q_n - p_n) \). \( d(x_n, p_n) = |t_n| d(q_n, p_n) \) approaches to \( |t_n| d(q, p) (n \to \infty) \), while \( x_n \) converges to \( x \) as \( p_n, q_n \) and \( t_n \) approach to \( p, q \) and \( t \) respectively.

Hence there exists \( d(x, p) \) provided that we take \( t_n \) by \( t_n = \frac{1}{2} - \frac{1}{n+1} \)

The proof is complete.

**Lemma 4.4.** If \( v_0, v_1, \ldots, v_k \) are the vertices of a star finite complex and \( \hat{S}(v_i) \) is the open stars of \( v_i (i = 0, 1, \ldots, k) \), then \( \langle v_0v_1\ldots v_k \rangle \subseteq K \) if and only if \( \bigcap_{i=0}^{k} \hat{S}(v_i) \neq \phi \).

Proof: \((\rightarrow)\) \( \forall v \in \langle v_0v_1\ldots v_k \rangle \subseteq K \), let \( h = \sum_{i=0}^{k} x_i v_i, \ \sum_{i=0}^{k} x_i - 1, \ \sum_{i=0}^{k} x_i v_i \), then there exists \( h \in \hat{S}(v_i) \).

Thus \( \bigcap_{i=0}^{k} \hat{S}(v_i) \neq \phi \).

\((\leftarrow)\) Let \( v_0, \ldots, v_k \) be any vertices, we can represent \( v_0, \ldots, v_k \) as positive. Hence \( \langle v_0v_1\ldots v_k \rangle \) forms a simplex of \( K \).

From the above lemmas, we note that if \( K \) and \( L \) are triangulations of the polytopes \( |K| \) and \( |L| \) respectively, and if \( f \) is a continuous mapping of \( |K| \) into \( |L| \), then \( K \) is
called star-related to $L$ relative to $f$. And for every $p_i$ of $K$, there is a vertex $v_i$ of $L$ such that $f(\hat{St}(p_i)) \subset \hat{St}(v_i)$ where $p_i$ and $v_i$ belong to $K$, $L$ respectively.

**Theorem 4.5.** Let both $|K|$ and $|L|$ be finite polytopes with triangulations $K$ and $L$ respectively, and if $f$ is a continuous mapping of $|K|$ into $|L|$. If $K$ is star-related to $L$ relative to $f$, then there exists a mapping $S$ of $|K|$ into $|L|$ such that

1. $s : K \to L$, simplicial mapping
2. $\forall x \in |K|$, $f(x) \in L$ such that both $f(x)$ and $s(x)$ belong to $\hat{St}(v_i)$.
3. $s$ : homotopic to $f$.

Proof: Assume $K_i = \{v | v : \text{vertex of } K\}$ and $L_i = \{v | v : \text{vertex of } L\}$. For all $p_i$ of $K$, there is $v_i(i)$ of $L_i$ such that $f(\hat{St}(p_i)) \subset \hat{St}(v_i(i))$. And then we define a correspondence $s$ between the vertices of $K$ and those of $L$ by setting

$s(p_i) = v_i(i)$.

Then for all $p_i$ of $K$, there exists $v_i(i) = s(p_i)$ of $L$ and for all $\langle p_0 p_1 \cdots p_s \rangle$ of $K_i$, $\bigcap_{i=0}^s \hat{St}(p_i)$ is not empty by Lemma 4.4.

Since $f \left( \bigcap_{i=0}^s \hat{St}(p_i) \right) \subset \bigcap_{i=0}^s f(\hat{St}(p_i)) \subset \bigcap_{i=0}^s \hat{St}(v_i(i)) = \bigcap_{i=0}^s \hat{St}(s(p_i))$ is not empty.

Again from Lemma 4.4, $\langle s(p_0) \cdots s(p_s) \rangle$ is a simplex of $L$. Thus $s$ is simplicial, and by barreymetric extension we obtain a continuous mapping of $|K|$ into $|L|$.

To show that (2) succeed, every point $x$ in $|K|$ lies in the interior of some simplex $s^*$ of $K$, $s^*$ taken to be of minimum dimension, i.e., $s^* = \langle p_0 p_1 \cdots p_s \rangle = x = \sum_{i=0}^s x_i p_i$, $\sum_{i=0}^s x_i = 1$, $x_i > 0$.

If $p$ is any vertex of $s^*$, then $x$ lies in $\hat{St}(p)$. By definition 4.1, $f(x) \in f(\hat{St}(p)) \subset \hat{St}(s(p))$ i.e. $f(x) \in \hat{St}(s(p))$. But also $s(x)$ lies in $\hat{St}(s(p))$, since $s(x) = \sum_{i=0}^s x_i s(p_i)$,

$\sum_{i=0}^s x_i = 1$, and $x_i > 0$.

Thus the mapping satisfies condition (2).

It remains to show that $s$ is homotopic to $f$. we define $h$ by $|K| \times I^1 \to |L|$ such that $h(x, 0) = f(x)$ and $h(x, 1) = s(x)$, for each $x$ of $|K|$. Let $x$ be a point of $s^* = \langle p_0 \cdots p_s \rangle$ in $K$. Since $s$ is simplicial, $s(p_0), \ldots, s(p_s)$ are vertices of a simplex $s^*$ in $L$.

Having that $f(x)$ lies in $\hat{St}(s(p_i))$ for each vector of $s^*$, it follows that $f(x)$ is a point of $\bigcap_{i=0}^s \hat{St}(s(p_i))$, which is precisely the simplex $s^*$ of $L$. Having both $f(x)$ and $s(x)$ in the same simplex $s^*$ of $L$, we make use of the convexity of $s^*$ and join $f(x)$ to $s(x)$ by a (unique) line segment in $s^*$.

Properly metrized, this line segment will be the image under a homotopy $h$ of the line
segment $x \times I$ in the homotopy cylinder $|K| \times I$, Letting $d(f(x), s(x)) = 1$, we write in vector notation,

$$h(x, t) = (1-t)f(x) + ts(x)$$

The continuity of $h$ as defined here is a consequence of Lemma 4.3.

5. Conclusion

We have looked at a special class of continuous mappings of one polytope into another, namely, those mappings which carry simplexes linearly onto simplexes.

With the aid of Lemma 4.2, 4.3 and 4.4, we have completed the proof of theorem 4.5.

From the present process, we see that it is shown the simplicial approximation theorem [5] if we next replace the triangulations $K$ and $L$ by barycentric subdivisions $K^*$ and $L^*$, $L^*$ being chosen to yield the desired accuracy of approximation and $K^*$ being chosen so as to be star related to $L^*$ relative to $I$.

References
