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PURELY INFINITE CUNTZ–KRIEGER ALGEBRAS OF DIRECTED GRAPHS

JEONG HEE HONG AND WOJCIECH SZYMAŃSKI

ABSTRACT

For arbitrary infinite directed graphs E, the characterisation of the (not necessarily simple) Cuntz–Krieger algebras $C^*(E)$ which are purely infinite in the sense of Kirchberg–Rørdam is given. It is also shown that $C^*(E)$ has real rank zero if and only if the graph E satisfies Condition (K).

Introduction

Purely infinite simple C*-algebras were first defined and investigated by Cuntz [**4**, **5**]. They have the characteristic property that for every non-zero element x there exist a, b such that $axb = 1$. A large class of examples of such algebras is provided by simple Cuntz–Krieger algebras [**6**]. In two recent articles [**16**, **17**], Kirchberg and Rørdam extended the concept of pure infiniteness to the case of non-simple C*-algebras. According to [**16**, Definition 4.1], a C*-algebra A is purely infinite if it has no characters and if, for every pair of positive elements x, y in A such that y lies in the closed two-sided ideal generated by x, there exists a sequence $a_n \in A$ such that $a_n^* x a_n \to y$. If A is simple, then the Kirchberg–Rørdam definition agrees with that of Cuntz. It is the purpose of this note to give a convenient characterization of generalized Cuntz–Krieger algebras based on directed graphs that are purely infinite in the sense of Kirchberg–Rørdam.

Quite recently, the theory of graph algebras has been developed by a number of researchers (see [**1**, **2**, **7**, **9**, **18**, **19**, **20**, **21**], among others) in an attempt to produce a far-reaching and yet accessible generalization of the Cuntz–Krieger algebras of finite matrices. Indeed, graph algebras do provide a large and interesting class of examples of C*-algebras, both simple and non-simple ones. Questions related to infiniteness entered their theory almost from the beginning. However, in most of the previous papers on the subject, pure infiniteness of a C*-algebra A was understood as the property that every non-zero hereditary subalgebra of A should possess an infinite projection. This coincides with Cuntz's definition for simple C*-algebras, but disagrees with the Kirchberg–Rørdam definition for many non-simple C*-algebras. For example, the minimal unitization of the stable $\mathcal{O}_2$ has this property, but is not purely infinite in the sense of Kirchberg–Rørdam [**16**, Example 4.6]. (Note that the minimal unitization of the stable $\mathcal{O}_2$ is an example of a graph algebra; the corresponding graph has two vertices v, w, two edges with sources and ranges at w, and infinitely many edges from v to w.) The key result states that if E is a countable directed graph, then every non-zero hereditary subalgebra of $C^*(E)$ has an infinite projection if and only if the following two conditions are satisfied: (i) every loop in E has an exit, and (ii) every vertex connects to a loop by an oriented path.

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This result was originally proved for locally finite graphs without sinks in [**18**, Theorem 3.9]. (An alternative proof was given in [**15**, Theorem 5.1].) It was then extended to arbitrary row-finite graphs in [**2**, Proposition 5.3 and Remark 5.5], and proved in full generality in [**7**, Corollary 2.14]. Analogous results were also obtained for other, but related, classes of generalized Cuntz–Krieger algebras in [**8**, Theorem 16.2] and [**22**, Theorem 4.4].

Pure infiniteness of graph algebras in the sense of Kirchberg–Rørdam was considered by Hjelmborg in [**10**], and by Jeong and Park in [**14**]. In both these papers, only locally finite graphs are studied; that is, graphs such that each vertex emits and receives only finitely many edges. For such graphs, a number of conditions equivalent to pure infiniteness of $C^*(E)$ is given in [**10**, Theorem 3.1] and [**14**, Corollary 4.1]. Our main result, Theorem 2.3, gives several such conditions for an arbitrary graph E. The proof relies heavily on the classification of gauge-invariant ideals of $C^*(E)$ from [**1**]. In particular, we show that all purely infinite graph algebras satisfy the stronger condition that every nonzero hereditary C*-subalgebra in every quotient contains an infinite projection. Furthermore, we show that all such graph algebras have real rank zero. We then characterize the C*-algebras $C^*(E)$ with real rank zero as those for which E satisfies Condition (K) of [**19**], thus extending the analogous result for locally finite graphs of Jeong and Park [**14**, Theorem 4.1]. After this paper had been submitted, we learnt that Jeong had obtained a different proof of Theorem 2.5; see [**13**].

1. *Preliminaries on graph algebras*

We recall the definition of the C*-algebra corresponding to a directed graph [**9**]. Let $E = (E^0, E^1, r, s)$ be a directed graph with countably many vertices E^0 and edges E^1, and range and source functions $r, s : E^1 \to E^0$, respectively. $C^*(E)$ is defined as the universal C*-algebra generated by families of projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$, subject to the following relations:

(GA1) $P_v P_w = 0$ for $v, w \in E^0$, $v \neq w$;
(GA2) $S_e^* S_f = 0$ for $e, f \in E^1$, $e \neq f$;
(GA3) $S_e^* S_e = P_{r(e)}$ for $e \in E^1$;
(GA4) $S_e S_e^* \leqslant P_{s(e)}$ for $e \in E^1$;
(GA5) $P_v = \sum_{e \in E^1 : s(e)=v} S_e S_e^*$ for $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$.

'Universality' in this definition means that if $\{Q_v : v \in E^0\}$ and $\{T_e : e \in E^1\}$ are families of projections and partial isometries, respectively, satisfying conditions (GA1)–(GA5), then there exists a C*-algebra homomorphism from $C^*(E)$ to the C*-algebra generated by $\{Q_v : v \in E^0\}$ and $\{T_e : e \in E^1\}$ such that $P_v \mapsto Q_v$ and $S_e \mapsto T_e$ for $v \in E^0$, $e \in E^1$.

It follows from the universal property that there exists a gauge action $\gamma : \mathbb{T} \to \mathrm{Aut}(C^*(E))$ such that $\gamma_t(P_v) = P_v$ and $\gamma_t(S_e) = tS_e$ for all $v \in E^0$, $e \in E^1$, $t \in \mathbb{T}$.

If $\alpha_1, \ldots, \alpha_n$ are (not necessarily distinct) edges such that $r(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \ldots, n-1$, then $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a path of length $|\alpha| = n$, with source $s(\alpha) = s(\alpha_1)$ and range $r(\alpha) = r(\alpha_n)$. A *loop* is a path of positive length whose source and range coincide. A loop α *has an exit* if there exist an edge $e \in E^1$ and an index i such that $s(e) = s(\alpha_i)$ but $e \neq \alpha_i$. If α is a loop all of whose vertices belong to a subset $M \subseteq E^0$, then we say that α *has an exit in* M if an edge e exists as above with $r(e) \in M$. A graph is said to *satisfy Condition* (K) if every vertex $v \in E^0$ lies on no loops, or if there are two loops α and μ such that $s(\alpha) = s(\mu) = v$ and neither

α nor μ is an initial subpath of the other [**19**]. A graph is *row-finite* if every vertex emits only finitely many edges.

By an *ideal* in a C*-algebra we always mean a closed two-sided ideal. An ideal J is called *gauge-invariant* if $\gamma_t(J) \subseteq J$ for all $t \in \mathbb{T}$. In order to understand the ideal structure of a graph algebra, it is convenient to look at saturated hereditary subsets of the vertex set. As usual, if $v, w \in E^0$, then we write $v \geqslant w$ when there is a path from v to w, and we say that a subset K of E^0 is *hereditary* if $v \in K$ and $v \geqslant w$ imply that $w \in K$. A subset K of E^0 is *saturated* if every vertex v that satisfies $0 < |s^{-1}(v)| < \infty$ and $s(e) = v \Longrightarrow r(e) \in K$ itself belongs to K. If $X \subseteq E^0$, then $\Sigma(X)$ is the smallest saturated subset of E^0 containing X, and $\Sigma H(X)$ is the smallest saturated hereditary subset of E^0 containing X. If K is hereditary and saturated, then I_K denotes the ideal of $C^*(E)$ generated by $\{P_v : v \in K\}$. As shown in [**1**, Proposition 3.4], the quotient $C^*(E)/I_K$ is naturally isomorphic to the graph algebra $C^*(E/K)$. The *quotient graph* E/K was defined in [**1**, Section 3]. The vertices of E/K are

$$(E^0 \setminus K) \cup \{\beta(v) : v \in K_\infty^{\text{fin}}\},$$

where

$$K_\infty^{\text{fin}} = \{v \in E^0 \setminus K : |s^{-1}(v)| = \infty \text{ and } 0 < |s^{-1}(v) \cap r^{-1}(E^0 \setminus K)| < \infty\}.$$

The edges of E/K are

$$r^{-1}(E^0 \setminus K) \cup \{\beta(e) : e \in E^1, r(e) \in K_\infty^{\text{fin}}\},$$

with the source and range functions extended by

$$s(\beta(e)) = s(e) \quad \text{and} \quad r(\beta(e)) = \beta(r(e)),$$

respectively. Note that all extra vertices $\beta(K_\infty^{\text{fin}})$ are sinks in E/K. If $K_\infty^{\text{fin}} = \emptyset$, then E/K is simply a subgraph of E (the restriction of E to $E^0 \setminus K$), and is hence denoted $E \setminus K$. If $v \in K_\infty^{\text{fin}}$, then we write

$$P_{v,K} = \sum_{s(e)=v,\, r(e)\notin K} S_e S_e^*.$$

For $B \subseteq K_\infty^{\text{fin}}$, the ideal of $C^*(E)$ generated by I_K and $\{P_v - P_{v,K} : v \in B\}$ is denoted by $J_{K,B}$. By [**1**, Corollary 3.5], the quotient $C^*(E)/J_{K,B}$ is naturally isomorphic to $C^*((E/K) \setminus \beta(B))$. Clearly, every ideal $J_{K,B}$ is gauge-invariant, since it is generated by projections fixed by the gauge action. It is shown in [**1**, Theorem 3.6] that all gauge-invariant ideals of $C^*(E)$ are of the form $J_{K,B}$.

In the discussion of primitive gauge-invariant ideals it is useful to use the following notation. For $X \subseteq E^0$, $\Omega(X)$ is defined as the collection of vertices $w \in E^0 \setminus X$ such that there is no path from w to any vertex in X. That is,

$$\Omega(X) = \{w \in E^0 \setminus X : w \not\geqslant v \text{ for all } v \in X\}.$$

A non-empty subset $M \subseteq E^0$ is a *maximal tail* (see [**1**, Lemma 4.1]) if it satisfies the following three conditions.

(MT1) If $v \in E^0$, $w \in M$, and $v \geqslant w$, then $v \in M$.

(MT2) If $v \in M$ and $0 < |s^{-1}(v)| < \infty$, then there exists $e \in E^1$ with $s(e) = v$ and $r(e) \in M$.

(MT3) For every $v, w \in M$ there exists $y \in M$ such that $v \geqslant y$ and $w \geqslant y$.

If M is a maximal tail, then $\Omega(M) = E^0 \setminus M$. We denote by $\mathscr{M}(E)$ the collection of all maximal tails in E. However, not all maximal tails give rise to gauge-invariant ideals, but only those in which every loop has an exit. We denote the set of all such maximal tails in E by $\mathcal{M}_\gamma(E)$. We employ an analogous notation, $\mathrm{Prim}_\gamma(C^*(E))$, to denote the set of all primitive gauge-invariant ideals in $C^*(E)$.

A $v \in E^0$ is a *breaking vertex* if $0 < |s^{-1}(v) \setminus r^{-1}(\Omega(v))| < \infty$ and $|s^{-1}(v)| = \infty$. The collection of all breaking vertices is denoted $\mathrm{BV}(E)$. If a vertex v emits infinitely many edges, then $\Omega(v)$ is automatically saturated and hereditary. In particular, if $v \in \mathrm{BV}(E)$, then $\Omega(v)$ is saturated and hereditary. Note that if $K \subseteq E^0$ is hereditary and saturated and $v \in K_\infty^{\mathrm{fin}}$, then $v \in \mathrm{BV}(E)$. As shown in [**1**, Theorem 4.7], there is a one-to-one correspondence between $\mathcal{M}_\gamma(E) \cup \mathrm{BV}(E)$ and $\mathrm{Prim}_\gamma(C^*(E))$, given by

$$\mathcal{M}_\gamma(E) \ni M \longleftrightarrow J_{\Omega(M),\Omega(M)_\infty^{\mathrm{fin}}} \in \mathrm{Prim}_\gamma(C^*(E));$$
$$\mathrm{BV}(E) \ni v \longleftrightarrow J_{\Omega(v),\Omega(v)_\infty^{\mathrm{fin}} \setminus \{v\}} \in \mathrm{Prim}_\gamma(C^*(E)).$$

2. *Purely infinite graph algebras*

The following simple result belongs to the folklore in the field, although we cannot recall its explicit statement in the literature. In the context of the Cuntz–Krieger algebras of finite matrices, it goes back to [**11**], and it is contained in the results of [**12**]. For the sake of completeness, we include the proof.

LEMMA 2.1. *Let E be a directed graph. If there is a loop without exits in E, then $C^*(E)$ contains an ideal that is not invariant under the gauge action.*

Proof. Let v be a vertex that lies on a loop without exits in E. Then the ideal I_v of $C^*(E)$ generated by P_v is Morita equivalent to $C(\mathbb{T})$; see [**18**]. So it suffices to show that the only gauge-invariant ideals of $C^*(E)$ contained in I_v are $\{0\}$ and I_v itself. Let J be such an ideal. If $J \neq \{0\}$, then there is a $w \in E^0$ such that $P_w \in J$, by the gauge-invariant uniqueness theorem [**1**, Theorem 2.1]. We have $w \in \Sigma H(v)$ by [**1**, Lemma 3.2]. Since $\Sigma H(v) = \Sigma(v)$, there is a path from w to v. Thus $P_v \in J$, and consequently $J = I_v$. □

Before proving our main result, Theorem 2.3, we need the following Lemma 2.2. The implication (a)⇒(c) of this lemma is already known (see [**2**] for the row-finite case, and [**7**] for the general case).

LEMMA 2.2. *If E is a directed graph, then the following conditions are equivalent.*
(a) *The graph E satisfies Condition* (K).
(b) *All loops in each maximal tail M have exits in M.*
(c) *Every ideal of $C^*(E)$ is gauge-invariant.*

Proof. (a) ⇒ (b). This is obvious.

(b) ⇒ (c). The proof of this implication is almost identical with that of [**1**, Corollary 3.8]. We give it here for the sake of completeness.

It suffices to show that every primitive ideal J of $C^*(E)$ is gauge-invariant. For such a J, let $K = \{v \in E^0 : P_v \in J\}$ and $B = \{v \in K_\infty^{\mathrm{fin}} : P_v - P_{v,K} \in J\}$. We have $J_{K,B} \subseteq J$. Both quotients $C^*(E)/J_{K,B}$ and $C^*(E)/J$ are generated by Cuntz–Krieger $((E/K) \setminus \beta(B))$-families in which all projections associated to vertices are different from zero. The set $M = E^0 \setminus K$ is a maximal tail by [**1**, Lemma 4.1], and hence every loop in M has an exit in M, by hypothesis. It follows that all the loops in

$(E/K) \setminus \beta(B)$ have exits. Thus, two applications of the Cuntz–Krieger uniqueness theorem [**20**, Theorem 1.5] show that both quotients $C^*(E)/J_{K,B}$ and $C^*(E)/J$ are canonically isomorphic to $C^*((E/K) \setminus \beta(B))$. Consequently, the quotient map of $C^*(E)/J_{K,B}$ onto $C^*(E)/J$ is an isomorphism, and $J = J_{K,B}$ is gauge-invariant.

(c) $\Rightarrow$ (a). Suppose that the graph E does not satisfy Condition (K). Then there exists a vertex $v \in E^0$ and a loop α with $s(\alpha) = v$ such that α is an initial subpath of any other loop μ with $s(\mu) = v$. It follows that $\Omega(v)$ is a saturated hereditary subset of E^0 and $v \notin \Omega(v)$. By [**1**, Corollary 3.5], the quotient $C^*(E)/J_{\Omega(v),\Omega(v)^{\mathrm{fin}}_\infty}$ is canonically isomorphic with $C^*(E \setminus \Omega(v))$. Note that α is a loop without exits in the graph $E \setminus \Omega(v)$. Indeed, if $e \in E^1$ is an exit for α, then $r(e) \in \Omega(v)$, since E does not satisfy Condition (K). Thus $C^*(E \setminus \Omega(v))$ contains an ideal that is not invariant under the gauge action, by Lemma 2.1. Therefore $C^*(E)$ contains such an ideal as well. □

THEOREM 2.3. *If E is a directed graph, then the following conditions are equivalent.*

(a) *Every nonzero hereditary C*-subalgebra in every quotient of $C^*(E)$ contains an infinite projection.*

(b) *The C*-algebra $C^*(E)$ is purely infinite in the sense of Kirchberg–Rørdam.*

(c) *There are no breaking vertices in E, and for each vertex $v \in E^0$ the projection P_v is properly infinite in $C^*(E)$.*

(d) *There are no breaking vertices in E, all loops in each maximal tail M have exits in M, and each vertex in each maximal tail M connects to a loop in M.*

(e) *There are no breaking vertices in E, the graph E satisfies Condition* (K), *and each vertex in each maximal tail M connects to a loop in M.*

Proof. (a) $\Rightarrow$ (b). This is [**16**, Proposition 4.7].

(b) $\Rightarrow$ (c). If $C^*(E)$ is purely infinite, then each projection P_v is properly infinite, by [**16**, Theorem 4.16]. If $v \in E^0$ is a breaking vertex, then $v \in \Omega(v)^{\mathrm{fin}}_\infty$. We have

$$C^*(E)/J_{\Omega(v),\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\}} \cong C^*((E/\Omega(v)) \setminus \beta(\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\})),$$

and $\beta(v)$ is a sink in the graph $(E/\Omega(v)) \setminus \beta(\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\})$. Thus, the corresponding projection $P_v - P_{v,\Omega(v)}$ (see [**1**, Proposition 3.4]) in $C^*(E)/J_{\Omega(v),\Omega(v)^{\mathrm{fin}}_\infty \setminus \{v\}}$ generates an ideal isomorphic with the compacts, contradicting [**16**, Proposition 4.3].

(c) $\Rightarrow$ (d). Since there are no breaking vertices in E, for each maximal tail M we have $\Omega(M)^{\mathrm{fin}}_\infty = \emptyset$, and hence $C^*(E)/I_{\Omega(M)} \cong C^*(M)$, by [**1**, Proposition 3.4].

Suppose that M is a maximal tail in E, and that α is a loop without exits in M. Then the ideal of $C^*(M)$ generated by $P_{s(\alpha)}$ is Morita equivalent to $M_{|\alpha|}(\mathbb{C}) \otimes C(\mathbb{T})$ (see [**18**]). Thus $P_{s(\alpha)}$ is not properly infinite in $C^*(E)/I_{\Omega(M)}$, and hence it is not properly infinite in $C^*(E)$ either, a contradiction.

Suppose that M is a maximal tail in E, that $v \in M$, and that there is no path from v to a loop in M. Let H be the set of those vertices $w \in M$ such that there exists a path from v to w. Then H is a hereditary subset of M containing v, and H does not contain any loop. The ideal I_v of $C^*(M)$ generated by P_v is Morita-equivalent to $C^*(H)$ (see [**18**]) and hence it is an AF-algebra since H has no loops (see [**18**, Theorem 2.4] and [**20**, Remark 5.4]). Thus the projection P_v is not properly infinite in $C^*(E)/I_{\Omega(M)}$, and hence it is not properly infinite in $C^*(E)$ either, a contradiction.

(d) $\Rightarrow$ (e). This follows from Lemma 2.2.

(e) $\Rightarrow$ (a). Every ideal of $C^*(E)$ is gauge-invariant by Lemma 2.2, and hence it has the form $J_{K,B}$ for some saturated hereditary $K \subseteq E^0$ and $B \subseteq K^{\mathrm{fin}}_\infty$ by

[1, Theorem 3.6]. We may assume that $K \neq E^0$, for otherwise $J_{K,B} = C^*(E)$. Since $\mathrm{BV}(E) = \varnothing$, we have $K_\infty^{\mathrm{fin}} = \varnothing$ as well. Thus $J_{K,B} = I_K$ and the quotient $C^*(E)/I_K$ is canonically isomorphic with $C^*(E \setminus K)$, by [1, Proposition 3.4].

We claim that in the graph $E \setminus K$: (i) each loop has an exit, and (ii) each vertex connects to a loop. Indeed, let α be a loop in $E \setminus K$. Then, by Condition (K), there is a loop μ in E with $s(\mu) = s(\alpha)$ such that neither α nor μ is an initial subpath of the other. Since K is hereditary, all vertices of μ must lie in $E^0 \setminus K$, and hence μ gives rise to an exit in $E \setminus K$ for α. This proves claim (i). Let $v \in E^0 \setminus K$. There exists a primitive ideal J of $C^*(E \setminus K)$ such that $P_v \notin J$. Since $\mathrm{BV}(E) = \varnothing$, it follows from [1, Theorem 4.7] that there exists a maximal tail M in $E \setminus K$ such that $v \in M$. Clearly, M is also a maximal tail in E. Thus, by hypothesis, there exists a path from v to a loop in M. This proves claim (ii).

It now follows from [7, Corollary 2.14] that each non-zero hereditary C*-subalgebra of $C^*(E \setminus K)$ contains an infinite projection. □

Recall that, by definition, a unital C*-algebra A has real rank zero (denoted $\mathrm{RR}(A) = 0$) if and only if invertible self-adjoint elements of A are dense in the set of all self-adjoint elements of A; see [3]. The real rank of a non-unital algebra A is by definition the real rank of the minimal unitization of A. By [3, Theorem 2.6], $\mathrm{RR}(A) = 0$ if and only if the self-adjoint elements of A with finite spectra are dense in the set of all self-adjoint elements of A. In [14, Theorem 4.1], Jeong and Park proved that for a locally finite graph E, Condition (K) is equivalent to saying that $\mathrm{RR}(C^*(E)) = 0$. The same fact remains true for arbitrary graphs, as Theorem 2.5 shows. (A different proof of this result is given in [13].) To prove the theorem we need the following lemma, which is of independent interest.

LEMMA 2.4. *Let E be a directed graph satisfying Condition* (K). *Then there exists an increasing sequence of C*-subalgebras A_n of $C^*(E)$ such that the closure of the union of all A_n equals $C^*(E)$, and there exists a sequence of finite graphs F_n satisfying Condition* (K) *such that for each n, the C*-algebras A_n and $C^*(F_n)$ are isomorphic.*

Proof. At first we associate with each vertex $u \in E^0$ a subgraph $G(u)$ of E, as follows. If there is no loop in E passing through u, then $G(u)$ consists of the single vertex u and no edges. Otherwise, since E satisfies Condition (K), there exist two loops α and μ such that $s(\alpha) = s(\mu) = u$ and neither α nor μ is an initial subpath of the other. Then the subgraph $G(u)$ of E consists of all the vertices and all the edges of two such loops.

Now we enumerate the vertices $E^0 = \{v_n : n = 1, 2, \ldots\}$ and the edges $E^1 = \{e_n : n = 1, 2, \ldots\}$ of E, and we construct by induction a sequence of subgraphs $\tilde{F}_n$, $n = 0, 1, \ldots$, of E, as follows. $\tilde{F}_0$ is defined as the empty graph. Suppose that the graph $\tilde{F}_{n-1}$ has already been defined. We define $\tilde{F}_n$ as the union of F_{n-1}, the edge e_n, and all $G(u)$ with $u \in F_{n-1}^0 \cup \{v_n, s(e_n), r(e_n)\}$. This definition implies that for each vertex w in $\tilde{F}_n$ there exists a vertex u such that $w \in G(u)$ and $G(u)$ is a subgraph of $\tilde{F}_n$. Therefore, if there is a loop in E passing through w, then there are at least two such loops in $\tilde{F}_n$. Consequently, each graph $\tilde{F}_n$ satisfies Condition (K).

We define A_n as the C*-subalgebra of $C^*(E)$ generated by the projections $\{P_v : v \in \tilde{F}_n^0\}$ and the partial isometries $\{S_e : e \in \tilde{F}_n^1\}$. Then A_n is isomorphic to a C*-algebra of a finite graph (see [20, Definition 1.1, Lemma 1.2 and Remark 5.1]), and the analysis of the construction in [20] shows that graph satisfies Condition (K). However, for the sake of completeness, we give another proof of this fact. Indeed, let V_n be the set of those vertices v of $\tilde{F}_n$ that emit at least one edge in $\tilde{F}_n$ and for which there exists an edge in E with source at v that does not belong to $\tilde{F}_n$. We define F_n as the graph $\tilde{F}_n$ enlarged by extra vertices $\{\bar{v} : v \in V_n\}$ and extra edges $\{\bar{e} : e \in \tilde{F}^1_n, r(e) \in V_n\}$, with the source and the range functions of $\tilde{F}_n$ extended by $s(\bar{e}) = s(e)$ and $r(\bar{e}) = \overline{r(e)}$, respectively. Since $\tilde{F}_n$ satisfies Condition (K) and all the additional vertices $\{\bar{v} : v \in V_n\}$ of F_n are sinks, it follows that F_n satisfies Condition (K) as well. It remains to prove that the C*-algebras A_n and $C^*(F_n)$ are isomorphic. Indeed, let $\{Q_v : v \in \tilde{F}^0_n\}$, $\{Q_{\bar{v}} : v \in V_n\}$, $\{T_e : e \in \tilde{F}^1_n\}$ and $\{T_{\bar{e}} : e \in \tilde{F}^1_n, r(e) \in V_n\}$ be the generators of $C^*(F_n)$. The map

$$Q_v \mapsto \begin{cases} P_v, & \text{if } v \text{ is a sink in } \tilde{F}_n, \\ \sum_{f \in \tilde{F}^1_n, s(f)=v} S_f S_f^*, & \text{otherwise,} \end{cases}$$

$$Q_{\bar{v}} \mapsto P_v - \sum_{f \in \tilde{F}^1_n, s(f)=v} S_f S_f^*,$$

$$T_e \mapsto \begin{cases} S_e, & \text{if } r(e) \text{ is a sink in } \tilde{F}_n, \\ S_e \sum_{f \in \tilde{F}^1_n, s(f)=r(e)} S_f S_f^*, & \text{otherwise,} \end{cases}$$

$$T_{\bar{e}} \mapsto S_e \Big(P_{r(e)} - \sum_{f \in \tilde{F}^1_n, s(f)=r(e)} S_f S_f^* \Big),$$

extends to a C*-algebra homomorphism from $C^*(F_n)$ to A_n since the target elements satisfy relations (GA1)–(GA5) for the graph F_n. This homomorphism is subjective, since the range contains the generators $\{P_v : v \in \tilde{F}^0_n\}$ and $\{S_e : e \in \tilde{F}^1_n\}$ of A_n. It is also injective, by the Cuntz–Krieger uniqueness theorem [2, Theorem 3.1], since Condition (K) implies that every loop in the graph F_n has an exit. Thus $A_n \cong C^*(F_n)$, and the lemma is proved. □

THEOREM 2.5. *If E is a directed graph, then $C^*(E)$ satisfies Condition* (K) *if and only if the real rank of $C^*(E)$ is zero.*

Proof. Suppose that E does not satisfy Condition (K). Then, as in the proof of the implication (c) ⇒ (a) of Lemma 2.2, we see that a quotient of $C^*(E)$ contains an ideal that is Morita equivalent to $C(\mathbb{T})$. Thus $\mathrm{RR}(C^*(E)) \neq 0$ by [3, Theorem 3.14 and Proposition 1.1].

Conversely, suppose that E satisfies Condition (K). Let A_n be a sequence of C*-subalgebras of $C^*(E)$, as in Lemma 2.4. Then $C^*(E)$ is the inductive limit $\varinjlim A_n$, and each A_n has real rank zero by [14, Theorem 4.1]. Thus $C^*(E)$ has real rank zero, by [3, Proposition 3.1]. □

Since all graph algebras are both separable (because E is a countable graph) and nuclear (see [20, Remark 4.3]), Theorems 2.3 and 2.5 and [17, Corollary 9.4] imply that the following corollary holds.

COROLLARY 2.6. *Let E be a directed graph. If $C^*(E)$ is purely infinite in the sense of Kirchberg–Rørdam, then* $\mathrm{RR}(C^*(E)) = 0$*; moreover, $C^*(E)$ and $C^*(E) \otimes \mathcal{O}_\infty$ are isomorphic as C*-algebras.*

References

1. T. Bates, J. H. Hong, I. Raeburn and W. Szymański, 'The ideal structure of the C*-algebras of infinite graphs', *Illinois J. Math.*, to appear.
2. T. Bates, D. Pask, I. Raeburn and W. Szymański, 'The C*-algebras of row-finite graphs', *New York J. Math.* 6 (2000) 307–324.
3. L. G. Brown and G. K. Pedersen, 'C*-algebras of real rank zero', *J. Funct. Anal.* 99 (1991) 131–149.
4. J. Cuntz, 'Simple C*-algebras generated by isometries', *Commun. Math. Phys.* 57 (1977) 173–185.
5. J. Cuntz, 'K-theory for certain C*-algebras', *Ann. of Math.* 113 (1981) 181–197.
6. J. Cuntz and W. Krieger, 'A class of C*-algebras and topological Markov chains', *Invent. Math.* 56 (1980) 251–268.
7. D. Drinen and M. Tomforde, 'The C*-algebras of arbitrary graphs'; math.OA/0009228.
8. R. Exel and M. Laca, 'Cuntz–Krieger algebras for infinite matrices', *J. Reine Angew. Math.* 512 (1999) 119–172.
9. N. J. Fowler, M. Laca and I. Raeburn, 'The C*-algebras of infinite graphs', *Proc. Amer. Math. Soc.* 128 (2000) 2319–2327.
10. J. v. B. Hjelmborg, 'Purely infinite and stable C*-algebras of graphs and dynamical systems', *Ergodic Theory Dynam. Systems* 21 (2001) 1789–1808.
11. A. an Huef, 'The Cuntz–Krieger algebras', Honours thesis, The University of Newcastle, 1993.
12. A. an Huef and I. Raeburn, 'The ideal structure of Cuntz–Krieger algebras', *Ergodic Theory Dynam. Systems* 17 (1997) 611–624.
13. J. A. Jeong, 'Real rank of generalized Cuntz–Krieger algebras', preprint, 2002.
14. J. A. Jeong and G. H. Park, 'Graph C*-algebras with real rank zero', *J. Funct. Anal.* 188 (2002) 216–226.
15. T. Kajiwara, C. Pinzari and Y. Watatani, 'Hilbert C*-bimodules and countably generated Cuntz–Krieger algebras', *J. Operator Theory* 45 (2001) 3–18.
16. E. Kirchberg and M. Rørdam, 'Non-simple purely infinite C*-algebras', *Amer. J. Math.* 122 (2000) 637–666.
17. E. Kirchberg and M. Rørdam, 'Infinite non-simple C*-algebras absorbing the Cuntz algebra $\mathcal{O}_\infty$', *Adv. Math.* 167 (2002) 195–264.
18. A. Kumjian, D. Pask and I. Raeburn, 'Cuntz–Krieger algebras of directed graphs', *Pacific J. Math.* 184 (1998) 161–174.
19. A. Kumjian, D. Pask, I. Raeburn and J. Renault, 'Graphs, groupoids, and Cuntz–Krieger algebras', *J. Funct. Anal.* 144 (1997) 505–541.
20. I. Raeburn and W. Szymański, 'Cuntz–Krieger algebras of infinite graphs and matrices', *Trans. Amer. Math. Soc.*, to appear.
21. W. Szymański, 'General Cuntz–Krieger uniqueness theorem', *Internat. J. Math.* 13 (2002) 549–555.
22. M. Tomforde, 'Simplicity of ultragraph algebras'; math.OA/0106162.

Jeong Hee Hong
Applied Mathematics
Korea Maritime University
Busan 606-791
South Korea

hongjh@hanara.kmaritime.ac.kr

Wojciech Szymański
Mathematics
The University of Newcastle
Callaghan, NSW 2308
Australia

wojciech@frey.newcastle.edu.au