VECTOR VARIATIONAL INEQUALITIES IN A HAUSDORFF TOPOLOGICAL VECTOR SPACE

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1. Introduction

Since Giannessi [5] first introduced a vector variational inequality problem (in short, VVIP) in an Euclidean space, VVIP has been intensively studied by many authors; for example, Chen [2], Chen and Li [3], Lee *et al.* [8,9], Lin [10], and Siddiqi *et al.* [13] (see also the references therein). In a series of recent papers, Yao *et al.* [7, 15] obtained two types of existence results of VVIP. To be more specific, in [15], Yu and Yao introduced the concept of weakly C-pseudomonotone operator. With this generalized monotonicity assumption, they provided several existence theorems on VVIP and applications to vector complementarity problem. On the other hand, in [7], Lai and Yao derived similar kind of existence results on VVIP without the generalized monotonicity assumption as a continuation of the previous work [15].

In this paper, we formulate more generalized versions of VVIP than Lai and Yao [7], and Yu and Yao [15], so we extend and sharpen two main theorems in [7, 15]. The point of generalization is to give, in a Hausdorff topological vector space, noncompact versions of the theorems under some coercivity condition in the case that domain X

is convex unbounded. Fan's lemma [4] has been the only one tool to prove almost all existence results on VVIP so far. But we use the Fan-Browder type fixed point theorem as a basic machinary to derive our results.

2. Preliminaries

Let E be a Hausdorff topological vector space and E^* its topological dual space. We say that E^* *separates points on* E provided for each $0 \neq x \in E$, there exists an $f \in E^*$ such that $\langle f, x \rangle \neq 0$. Here $\langle , \rangle$ denotes the usual pairing between E and E^*. A nonempty subset P of E is called *a convex cone* if

$$\lambda P \subset P, \quad \text{for all} \quad \lambda \geq 0 \quad \text{and} \quad P + P = P.$$

Let X be a nonempty convex subset of E, F another topological vector space and $C : X \to 2^F$ a multifunction such that for each $x \in X$, Cx is a convex cone in F with $\text{int}Cx \neq \emptyset$ and $Cx \neq F$, and $G : X \times X \to F$ a function. G is said to be
(1) *weakly C-pseudomonotone* if for any x, $y \in X$,

$$G(x, y) \notin -\text{int}Cx \quad \text{implies} \quad -G(y, x) \notin -\text{int}Cx; \text{ and}$$

(2) *v-hemicontinuous* if for any x, $y \in X$ and $t \in [0, 1]$, the map

$$t \mapsto G(x + t(y - x), y) \quad \text{is continuous at } 0^+.$$

We denote by $L(E, F)$ the space of all continuous linear mappings from E to F. Let $T : X \to L(E, F)$ be an operator. T is said to be
(1) *weakly C-pseudomonotone* if for any x, $y \in X$,

$$\langle Tx, y - x \rangle \notin -\text{int}Cx \quad \text{implies} \quad \langle Ty, y - x \rangle \notin -\text{int}Cx; \text{ and}$$

(2) *v-hemicontinuous* if for any x, $y \in X$ and $t \in [0, 1]$, the map

$$t \mapsto \langle T(x + t(y - x), y - x) \rangle \quad \text{is continuous at } 0^+.$$

The vector variational inequality problem is to find an $x \in X$ such that

$$\langle Tx, y - x\rangle \notin -\text{int}Cx \quad \text{for all} \quad y \in X.$$

Now we introduce a particular form of Park [11, Theorem 1] which is modified into convenient shape in order to derive main results. This theorem is a generalization of the well-known fixed point theorem of Fan-Browder [1, Theorem 1].

Theorem A. Let X be a nonempty convex subset of a Hausdorff topological vector space E, K a nonempty compact subset of X. Let $A,\ B : X \to 2^X$ be two multifunctions. Suppose that
(1) for each $x \in X,\ Ax \subset Bx$;
(2) for each $x \in X,\ Bx$ is convex;
(3) for each $x \in K,\ Ax$ is nonempty ;
(4) for each $y \in X,\ A^{-1}y$ is open ;
(5) for each finite subset N of X, there exists a nonempty compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K,\ Ax \cap L_N \neq \emptyset$. Then B has a fixed point x_0; that is, $x_0 \in Bx_0$.

3. Main Results

First we give the generalized linearization lemma as follows:

Lemma 3.1. Let $E,\ F$ be two Hausdorff topological vector spaces, X a nonempty convex subset of E. Let $C : X \to 2^F$ be a multifunction such that for each $x \in X,\ Cx$ is a convex cone in F with $\text{int}Cx \neq \emptyset$ and $Cx \neq F$, and $G : X \times X \to F$ a vector valued function. Define $P = \cap_{x \in X} Cx$ and consider the following problems:
(I) Find $x \in X$ such that $G(x, y) \notin -\text{int}Cx$ for all $y \in X$;
(II) Find $x \in X$ such that $-G(y, x) \notin -\text{int}Cx$ for all $y \in X$.

Then:

(i) Problem (I) implies Problem (II) if G is weakly C-pseudomonotone.

(ii) Problem (II) implies Problem (I) if the following conditions are satisfied;

(1) G is v-hemicontinuous;

(2) for each $x \in X$, $G(x, \cdot)$ is P-convex, that is, for any $y,\ z \in X$ and $\alpha \in [0,1]$, $G(x, \alpha y + (1-\alpha)z) \in \alpha G(x,y) + (1-\alpha)G(x,z) - P$; and

(3) for each $x \in X$, $G(x,x) \in P$.

Proof. (i) Let $x \in X$ be a solution of Problem (I). Then $G(x,y) \notin -\mathrm{int}Cx$ for all $y \in X$. Since G is weakly C-pseudomonotone, $-G(y,x) \notin -\mathrm{int}Cx$ for all $y \in X$. Hence, x is a solution of Problem (II).

(ii) Let $x \in X$ be a solution of Problem (II). Then we have

$$-G(y,x) \notin -\mathrm{int}Cx \quad \text{for all} \quad y \in X. \tag{3.1}$$

Suppose to the contrary that x is not a solution of Problem (I). Then there exists $\hat{y} \in X$ such that

$$G(x,\hat{y}) \in -\mathrm{int}Cx, \tag{3.2}$$

Let $x_t := x + t(\hat{y} - x)$ for $t \in [0,1]$. Since X is convex, $x_t \in X$. Also $G(x_t,\hat{y}) \to G(x,\hat{y})$ as $t \to 0^+$ because G is v-hemicontinuous. From (3.2), there exists a $\hat{t} \in (0,1]$ such that

$$G(x_t,\hat{y}) \in -\mathrm{int}Cx, \qquad \text{for all} \quad t \in (0,\hat{t}). \tag{3.3}$$

Fix $t \in (0,\hat{t})$. By the P-convexity of $G(x_t,\cdot)$, we have

$$G(x_t,x_t) = G(x_t, t\hat{y} + (1-t)x) \in tG(x_t,\hat{y}) + (1-t)G(x_t,x) - P.$$

From (3.3) and assumption (3), we have

$$\begin{aligned} -(1-t)G(x_t,x) &\in tG(x_t,\hat{y}) - G(x_t,x_t) - P \\ &\subset -\mathrm{int}C(x) - P - P \\ &\subset -\mathrm{int}Cx - Cx - Cx \\ &\subset -\mathrm{int}Cx. \end{aligned}$$

Hence $-G(x_t, x) \in -\text{int}C(x)$, which contradicts (3.1).

Remark. Lemma 3.1 is a generalization of the generalized linearization lemma in [15].

By Lemma 3.1, we obtain the following existence theorem of a vector inequality under the generalized monotonicity condition.

Theorem 3.1. Let E, F be two Hausdorff topological vector spaces, and let E^* separate points on E. Let X be a nonempty convex subset of E, and K a nonempty weakly compact subset of X. Let $C : X \to 2^F$ be a multifunction such that for each $x \in X$, Cx is a convex cone in F with $\text{int}Cx \neq \emptyset$ and $Cx \neq F$, and $G : X \times X \to F$ a function. Define $P = \cap_{x \in X} Cx$ and $W : X \to 2^F$, $Wx = F \setminus (-\text{int}Cx)$. The graph $Gr(W)$ of W is weakly closed in $X \times F$. Assume that the following conditions are satisfied :

(1) for each $x \in X$, $y \mapsto G(x, y)$ is weakly continuous and P-convex;
(2) G is weakly C-pseudomonotone and v-hemicontinuous;
(3) for each $x \in X$, $G(x, x) \in P$; and
(4) for each finite subset N of X, there exists a nonempty weakly compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there is a $y \in L_N$ satisfying $-G(y, x) \in -\text{int}Cx$.

Then there exists an $\bar{x} \in K$ such that $G(\bar{x}, x) \notin -\text{int}C\bar{x}$ for all $x \in X$.

Proof. Define two multifunctions A, $B : X \to 2^X$ to be

$$Ax = \{y \in X \mid -G(y, x) \in -\text{int}Cx\},$$
$$Bx = \{y \in X \mid G(x, y) \in -\text{int}Cx\}.$$

(i) By the weak C-pseudomonotonicity of G, $Ax \subset Bx$.
(ii) For each $x \in X$, Bx is convex. Indeed, when y, $z \in Bx$ and $\alpha \in [0, 1]$,

$$\begin{aligned} G(x, \alpha y + (1-\alpha)z) &\in \alpha G(x, y) + (1-\alpha)G(x, z) - P \\ &\subset \alpha(-\text{int}Cx) + (1-\alpha)(-\text{int}Cx) - P \\ &\subset -\text{int}Cx - Cx \\ &\subset -\text{int}Cx. \end{aligned}$$

Hence $\alpha y + (1-\alpha)z \in Bx$, as desired.
(iii) For each $y \in X$, $A^{-1}y$ is weakly open. In fact, let $\{x_\lambda\}$ be a net in $(A^{-1}y)^c$ weakly convergent to $x \in X$. Then $-G(y, x_\lambda) \notin -\mathrm{int}Cx_\lambda$, hence $-G(y, x_\lambda) \in Wx_\lambda$. Since $(x_\lambda, -G(y, x_\lambda)) \in Gr(W)$ and weakly converges to $(x, -G(y,x))$ by virtue of (1) and the weak closedness of $Gr(W)$, we have $-G(y,x) \in Wx$, i.e., $-G(y,x) \notin -\mathrm{int}Cx$. Thus $x \in (A^{-1}y)^c$. Therefore $(A^{-1}y)^c$ is weakly closed, hence $A^{-1}y$ is weakly open.
(iv) By the hypothesis (4), for each finite subset N of X, there exists a nonempty weakly compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there is a $y \in L_N$ satisfying $-G(y,x) \in -\mathrm{int}Cx$. Thus $Ax \cap L_N \neq \emptyset$.
(v) B has no fixed point. If not, there exists an $x \in X$ such that $G(x,x) \in -\mathrm{int}Cx$. By (3), $G(x,x) \in -\mathrm{int}Cx \cap P \subset -\mathrm{int}Cx \cap Cx = \emptyset$, a contradiction. Indeed, if there were a $v \in -\mathrm{int}Cx \cap Cx$, then $0 = -v + v \in \mathrm{int}Cx + Cx \subset \mathrm{int}Cx$. This implies $Cx = F$ because $\mathrm{int}Cx \ni 0$ is an absorbing set in F, which contradicts the assumption $Cx \neq F$. Therefore B has no fixed point.

From (i)-(v), we see, by Theorem A, that there must be an $\bar{x} \in K$ such that $A\bar{x} = \emptyset$, namely,

$$-G(y, \bar{x}) \notin -\mathrm{int}C\bar{x} \quad \text{for all} \quad y \in X.$$

Appealing to Lemma 3.1, we have

$$G(\bar{x}, y) \notin -\mathrm{int}C\bar{x} \quad \text{for all} \quad y \in X.$$

As a direct consequence of Theorem 3.1, we have the following.

Corollary 3.1. Let E, F, E^*, X, K, C, W, and P be the same as in Theorem 3.1. Let $T : X \to L(E,F)$ be weakly C-pseudomonotone and v-hemicontinuous. Assume that for each finite subset N of X, there exists a nonempty weakly compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there is a $y \in L_N$ satisfying $\langle Ty, y - x\rangle \in -\mathrm{int}Cx$. Then there exists an $\bar{x} \in K$ such that $\langle T\bar{x}, x - \bar{x}\rangle \notin -\mathrm{int}C\bar{x}$ for all $x \in X$.

Proof. Putting $G(x,y) = \langle Tx, y-x\rangle$ in Theorem 3.1, we get the result. Indeed, it is straightforward to check the conditions (1)-(4) of Theorem 3.1 except the weak continuity of $y \mapsto \langle Tx, y-x\rangle$ for each $x \in X$, in other words, the continuity of $Tx : (E,w) \to (F,w)$. But this directly follows from the definition of the weak topologies for E and F. (See Kelly and Namioka [6, 16.1 (iv) p.140]).

Remark. Corollary 3.1 is a noncompact generalization of Yu and Yao [15, Theorem 3.1] in a Hausdorff topological vector space E on which E^* separates points. They assumed E to be a Banach space. We used Fan-Browder type fixed point theorem as a basic tool to prove the existence of solution of VVIP whereas Yu and Yao [15] did Fan's lemma.

Now we provide an existence result of VVIP without the generalized monotonicity assumption.

Theorem 3.2. Let E, F, E^*, X, K, C, W, and P be the same as in Theorem 3.1. Let $G : X \times X \to F$ a function satisfying the following conditions:
(1) for each $x \in X$, $y \mapsto G(x,y)$ is P-convex;
(2) for each $y \in X$, $x \mapsto G(x,y)$ is weakly continuous;
(3) for each $x \in X$, $G(x,x) \in Cx$; and
(4) for each finite subset N of X, there exists a nonempty weakly compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there is a $y \in L_N$ satisfying $G(x,y) \in -\text{int}Cx$.

Then there exists an $\bar{x} \in K$ such that $G(\bar{x}, x) \notin -\text{int}C\bar{x}$ for all $x \in X$.

Proof. Define a multifunctions $A : X \to 2^X$ to be

$$Ax = \{y \in X \mid G(x,y) \in -\text{int}Cx\}.$$

(i) For each $x \in X$, Ax is convex and A has no fixed point as seen in the proof of Theorem 3.1.
(ii) For each $y \in X$, $A^{-1}y = \{x \in X \mid G(x,y) \in -\text{int}Cx\}$ is weakly open. In fact, let $\{x_\lambda\}$ be a net in $(A^{-1}y)^c$ weakly convergent to $x \in X$. Then $G(x_\lambda, y) \notin -\text{int}Cx_\lambda$, hence

$G(x_\lambda, y) \in Wx_\lambda$. Since $(x_\lambda, G(x_\lambda, y)) \in Gr(W)$ and weakly converges to $(x, G(x,y))$ by virtue of (2) and the weak closedness of $Gr(W)$, we have $G(x,y) \in Wx$, i.e., $G(x,y) \notin -\mathrm{int}Cx$. Thus $x \in (A^{-1}y)^c$. Therefore $(A^{-1}y)^c$ is weakly closed, namely, $A^{-1}y$ is weakly open.

(iii) By the hypothesis (4), for each finite subset N of X, there exists a nonempty weakly compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there is a $y \in L_N$ satisfying $G(x,y) \in -\mathrm{int}Cx$. Thus $Ax \cap L_N \neq \emptyset$.

From (i)-(iii), we see, by Theorem A, that there must be an $\bar{x} \in K$ such that $A\bar{x} = \emptyset$, namely,

$$G(\bar{x}, x) \notin -\mathrm{int}C\bar{x} \quad \text{for all} \quad x \in X.$$

Remark. Observe that the condition (3) of Theorem 3.1 is replaced by a weaker one "for each $x \in X$, $G(x,x) \in Cx$" in Theorem 3.2.

As an easy consequence of Theorem 3.2, we have the following.

Corollary 3.2. Let E, F, E^*, X, K, C, W, and P be the same as in Theorem 3.1. Let $T : X \to L(E,F)$ be a map satisfying $x \mapsto \langle Tx, y-x\rangle$ is weakly continuous. Assume that for each finite subset N of X, there exists a nonempty weakly compact convex subset L_N of X containing N such that for each $x \in L_N \setminus K$, there is a $y \in L_N$ satisfying $\langle Tx, y-x\rangle \in -\mathrm{int}Cx$. Then there exists an $\bar{x} \in K$ such that $\langle T\bar{x}, x-\bar{x}\rangle \notin -\mathrm{int}C\bar{x}$ for all $x \in X$.

Proof. Putting $G(x,y) = \langle Tx, y-x\rangle$ in Theorem 3.2, we get the result directly.

Remarks. (i) Corollary 3.2 is a noncompact generalization of Lai and Yao[7, Theorem 2.2] in a Hausdorff topological vector space E (not necessarily a normed or Banach space) on which E^* separates points. E^* is assumed to separate points on E so as to ensure that the weak topology for E is Hausdorff so that we can use Theorem A. The property that E^* separates points on E happens in every Hausdorff locally convex space (see Rudin [12, Corollary, p.59]). However, the converse is not true. Consider the metric space l^p, $0 < p < 1$. Then $(l^p)^*$ separates points on l^p but not locally convex space (see Rudin [12, Exercise 5 (d), p.82]).

(ii) Corollary 2.3 of Lai and Yao [7] may not be true. This is because they deduced it from the false fact that a weakly convergent net is strongly bounded in a Banach space. Of course, every weakly convergent sequence in a Banach space is strongly bounded by the Uniform Boundedness Principle (see Kelly and Namioka [6, Problem A, p.105]). However, as for a net, it is not sure. In addition, Corollaries 2.4 and 2.5 of Lai and Yao [7] may not be true because Corollaries 2.4 and 2.5 are deduced from Corollary 2.3.

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