벡터에 의한 球面三角法의 既存公式의 誘導에 관한 考察

A study of derivation of the known formulas by vectors in spherical trigonometry

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本稿는 벡터의 內積과 外積을 利用하여 球面三角法의 既存公式을 誘導하는 考察의 첫 階段이다.

球面三角形을 ABC 라 하고 頂点 A, B, C 에 對하는 邊 BC, CA, AB 의 크기를 a, b, c 라고 하면 球의 中心 O 를 A, B, C 와 맺어서 되는 三面角 $O-ABC$ 에서는 面角 BOC, COA, AOB 는 各各 a, b, c 가 된다.

球의 中心 O 에 對하는 A, B, C 의 位置벡터 $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ 를 各各 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ 라고 한다. 또 球의 半徑을 1 로 하고 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ 를 單位벡터로 看做한다.

이때 周知하는 바와 같이

$$\boldsymbol{a}\cdot\boldsymbol{b}=\cos c \qquad \boldsymbol{b}\cdot\boldsymbol{c}=\cos a \qquad \boldsymbol{c}\cdot\boldsymbol{a}=\cos b$$

$$|\boldsymbol{a}\times\boldsymbol{b}|=\sin c \qquad |\boldsymbol{b}\times\boldsymbol{c}|=\sin a \qquad |\boldsymbol{c}\times\boldsymbol{a}|=\sin b$$

가 된다. 또 $\boldsymbol{a}\times\boldsymbol{b}$, $\boldsymbol{a}\times\boldsymbol{c}$ 의 交角의 크기는 角 A 의 크기와 같고 $\boldsymbol{b}\times\boldsymbol{c}$, $\boldsymbol{b}\times\boldsymbol{a}$ 의 交角, $\boldsymbol{c}\times\boldsymbol{a}, \boldsymbol{c}\times\boldsymbol{b}$ 의 交角의 크기도 角 B, C 의 크기와 같다. 따라서

$$\left.\begin{aligned}\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{a}\times\boldsymbol{c}&=\sin c\sin b\cos A\\ \boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{b}\times\boldsymbol{a}&=\sin a\sin c\cos B\\ \boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{c}\times\boldsymbol{b}&=\sin b\sin a\cos C\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(\mathrm{I})$$

가 成立한다. 또 $(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}\times\boldsymbol{c})$ 의 方向은 $\boldsymbol{a}$ 의 方向과 같고 $(\boldsymbol{b}\times\boldsymbol{c})\times(\boldsymbol{b}\times\boldsymbol{a})$, $(\boldsymbol{c}\times\boldsymbol{a})\times(\boldsymbol{c}\times\boldsymbol{b})$ 의 方向도 各各 $\boldsymbol{b}, \boldsymbol{c}$ 의 方向과 같으므로

$$\left.\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}\times\boldsymbol{c})&=(\sin c\sin b\sin A)\boldsymbol{a}\\ (\boldsymbol{b}\times\boldsymbol{c})\times(\boldsymbol{b}\times\boldsymbol{a})&=(\sin a\sin c\sin B)\boldsymbol{b}\\ (\boldsymbol{c}\times\boldsymbol{a})\times(\boldsymbol{c}\times\boldsymbol{b})&=(\sin b\sin a\sin C)\boldsymbol{c}\end{aligned}\right\}\quad\cdots\cdots\cdots\cdots(\mathrm{II})$$

인 關係가 있다.

$\boldsymbol{a}\times\boldsymbol{b}$, $\boldsymbol{b}\times\boldsymbol{c}$, $\boldsymbol{c}\times\boldsymbol{a}$ 로서 두 벡터의 內積을 만들면

$$\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{b}\times\boldsymbol{c} \qquad \boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{c}\times\mathbf{a} \qquad \boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{a}\times\boldsymbol{b}$$

이고 이들에서

$$\left.\begin{aligned}\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{b}\times\boldsymbol{c}&=(\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{b}\cdot\boldsymbol{c})-\boldsymbol{a}\cdot\boldsymbol{c}\\ \boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{c}\times\boldsymbol{a}&=(\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{c}\cdot\boldsymbol{a})-\boldsymbol{b}\cdot\boldsymbol{a}\\ \boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{a}\times\boldsymbol{b}&=(\boldsymbol{c}\cdot\boldsymbol{a})(\boldsymbol{a}\cdot\boldsymbol{b})-\boldsymbol{c}\cdot\boldsymbol{b}\end{aligned}\right\}\cdots\cdots\cdots\cdots(\text{III})$$

가 된다. 이들에서

$$\left.\begin{aligned}-\sin c\sin a\cos B&=\cos c\cos a-\cos b\\ -\sin a\sin b\cos C&=\cos a\cos b-\cos c\\ -\sin b\sin c\cos A&=\cos b\cos c-\cos a\end{aligned}\right\}$$

가 誘導된다. 이것은 第 1 cosine 法則이다.

$\boldsymbol{a}\times\boldsymbol{b}$, $\boldsymbol{b}\times\boldsymbol{c}$, $\boldsymbol{c}\times\boldsymbol{a}$로서 두 벡터의 外積을 만들면

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{b}\times\boldsymbol{c})\qquad(\boldsymbol{b}\times\boldsymbol{c})\times(\boldsymbol{c}\times\boldsymbol{a})\qquad(\boldsymbol{c}\times\boldsymbol{a})\times(\boldsymbol{a}\times\boldsymbol{b})$$

이고 이들에서

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{b}-(\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{b})\boldsymbol{c}$$

即

$$(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{b}$$

또

$$\left.\begin{aligned}(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{b}\times\boldsymbol{c})&=(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{b}\\ (\boldsymbol{b}\times\boldsymbol{c})\times(\boldsymbol{c}\times\boldsymbol{a})&=(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{c}\\ (\boldsymbol{c}\times\boldsymbol{a})\times(\boldsymbol{a}\times\boldsymbol{b})&=(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{a}\end{aligned}\right\}\cdots\cdots\cdots\cdots(\text{IV})$$

이 成立한다. 이들에서

$$\left.\begin{aligned}(\sin c\sin a\sin B)\boldsymbol{b}&=(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{b}\\ (\sin a\sin b\sin C)\boldsymbol{c}&=(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{c}\\ (\sin b\sin c\sin A)\boldsymbol{a}&=(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})\boldsymbol{a}\end{aligned}\right\}$$

이고 따라서

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}&=\boldsymbol{b}\cdot\boldsymbol{c}\times\boldsymbol{a}=\boldsymbol{c}\cdot\boldsymbol{a}\times\boldsymbol{b}\\ &=\sin c\sin a\sin B=\sin a\sin b\sin C=\sin b\sin c\sin A\ \cdots\cdots\cdots(\text{V})\end{aligned}$$

인 關係가 成立하고 또 sine 法則

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}$$

가 誘導된다. 이 方法은 Spiegel 氏의 Vector analysis 에도 실여있다.

다음 演算이 行해지고

$$\begin{aligned}&\boldsymbol{a}\times\boldsymbol{b}\cdot(\boldsymbol{b}\times\boldsymbol{c})\times(\boldsymbol{c}\times\boldsymbol{a})\\ &=(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{c}\times\boldsymbol{a})-(\boldsymbol{b}\cdot\boldsymbol{b}\times\boldsymbol{c})(\boldsymbol{a}\cdot\boldsymbol{c}\times\boldsymbol{a})\\ &=(\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})^2\end{aligned}$$

따라서

$$\begin{aligned}&\boldsymbol{a}\times\boldsymbol{b}\cdot(\boldsymbol{b}\times\boldsymbol{c})\times(\boldsymbol{c}\times\boldsymbol{a})\\ &=(\boldsymbol{b}\times\boldsymbol{c})\cdot(\boldsymbol{c}\times\boldsymbol{a})\times(\boldsymbol{a}\times\boldsymbol{b})=\boldsymbol{c}\times\boldsymbol{a}\cdot(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{b}\times\boldsymbol{c})\end{aligned}$$

$$= (\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})^2 \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (\text{VI})$$

인 關係가 成立한다.

이 結果도 잘 알려진 것이다. 그러나 現在로서는 이들에서 球面三角法의 公式을 期待할 수는 없을 것 같다.

다음에

$$\left.\begin{aligned}
&(\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{c}\times\boldsymbol{a}\\
&\quad=(\boldsymbol{a}\cdot\boldsymbol{b})\{(\boldsymbol{a}\cdot\boldsymbol{b})(\mathbf{b}\cdot\mathbf{c})-\boldsymbol{a}\cdot\boldsymbol{c}\}+(\boldsymbol{a}\cdot\boldsymbol{c})(\boldsymbol{b}\cdot\boldsymbol{a})-\boldsymbol{b}\cdot\boldsymbol{c}\\
&(\boldsymbol{a}\cdot\boldsymbol{b})(\boldsymbol{a}\times\boldsymbol{b}\cdot\mathbf{c}\times\mathbf{a})+\boldsymbol{a}\times\boldsymbol{b}\cdot\mathbf{b}\times\boldsymbol{c}\\
&\quad=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{c}\times\boldsymbol{a})+\boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{a}\times\boldsymbol{b}\\
&\quad=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(\boldsymbol{b}\cdot\boldsymbol{c})(\boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{a}\times\boldsymbol{b})+\boldsymbol{b}\times\boldsymbol{c}\cdot\boldsymbol{c}\times\boldsymbol{a}\\
&\quad=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(\boldsymbol{c}\cdot\boldsymbol{a})(\boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{a}\times\boldsymbol{b})+\boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c}\\
&\quad=\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&(\boldsymbol{c}\cdot\boldsymbol{a})(\boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{b}\times\boldsymbol{c})+\boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{a}\times\boldsymbol{b}\\
&\quad=(\boldsymbol{c}\cdot\boldsymbol{a})\{(\boldsymbol{c}\cdot\boldsymbol{b})(\boldsymbol{a}\cdot\mathbf{c})-\boldsymbol{a}\cdot\boldsymbol{b}\}+(\boldsymbol{c}\cdot\boldsymbol{a})(\boldsymbol{a}\cdot\boldsymbol{b})-\boldsymbol{c}\cdot\mathbf{b}
\end{aligned}\right\}\cdots\cdots\cdots\cdots (\text{VII})$$

에서

$$\left.\begin{aligned}
&\cos c\,(-\sin a\sin c\cos B)-\sin c\sin b\cos A\\
&\quad=\cos c(\cos c\cos a-\cos b)+\cos b\cos c-\cos a\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cos b(-\sin b\sin a\cos C)-\sin c\sin b\cos A\\
&\quad=\cos b(\cos a\cos b-\cos c)+\cos b\cos c-\cos a
\end{aligned}\right\}$$

이 成立하고 이들을 簡單히 하면 차례로

$$\left.\begin{aligned}
\cos a\sin c&=\cos c\sin a\cos B+\sin b\cos A\\
\cos b\sin c&=\cos c\sin b\cos A+\sin a\cos B\\
\cos b\sin a&=\cos a\sin b\cos C+\sin c\cos B\\
\cos c\sin a&=\cos a\sin c\cos B+\sin b\cos C\\
\cos c\sin b&=\cos b\sin c\cos A+\sin a\cos C\\
\cos a\sin b&=\cos b\sin a\cos C+\sin c\cos A
\end{aligned}\right\}$$

를 얻는다. 이것은 第 3 cosine 法則이다.

다음에

$$\boldsymbol{a}'=\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\qquad \boldsymbol{b}'=\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\qquad \boldsymbol{c}'=\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}$$

가 球의 中心 O에 對하는 位置벡터가 되는 点을 A', B', C' 라고 하면 三角形 $A'B'C'$ 는 ABC 의 補三角形이 된다. 三角形 $A'B'C'$ 에서 頂点 A', B', C' 에 對하는 邊 $B'C', C'A', A'B'$ 를 a', b', c'

라고 하면 角 $B'OC', C'OA', A'OB'$ 는 各各 a', b', c' 가 된다.

이때

$$\begin{aligned}\sin(A+a') &= \sin A\cos a' + \cos A\sin a' \\ &= \frac{|(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}\times\boldsymbol{c})|}{\sin c\sin b}(\boldsymbol{b}'\cdot\boldsymbol{c}') + \frac{\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{a}\times\boldsymbol{c}}{\sin c\sin b}|\boldsymbol{b}'\times\boldsymbol{c}'| \\ &= \frac{|(\boldsymbol{a}\times\boldsymbol{b})\times(\boldsymbol{a}\times\boldsymbol{c})|}{\sin c\sin b}\,\frac{\boldsymbol{c}\times\boldsymbol{a}\cdot\boldsymbol{a}\times\boldsymbol{b}}{\sin b\sin c} + \frac{\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{a}\times\boldsymbol{c}}{\sin c\sin b}\,\frac{|(\boldsymbol{c}\times\boldsymbol{a})\times(\boldsymbol{a}\times\boldsymbol{b})|}{\sin b\sin c} \\ &= 0\end{aligned}$$

이므로 $A+a'=180°$가 된다. 같이하여

$$A+a'=B+b'=C+c'=A'+a=B'+b=C'+c=180°$$

가 成立할 것이다.

다음에 (Ⅲ)에서 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ 대신에 $\dfrac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}$, $\dfrac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}$, $\dfrac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}$를 代入하면

$$\left.\begin{aligned}&\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\times\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\times\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c} \\ &\quad=\left(\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\right)\left(\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\cdot\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\right)-\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c} \\ &\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\times\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\cdot\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\times\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a} \\ &\quad=\left(\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\cdot\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\right)\left(\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\cdot\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\right)-\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\cdot\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a} \\ &\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\times\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\times\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b} \\ &\quad=\left(\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\cdot\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\right)\left(\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\right)-\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\end{aligned}\right\}\cdots\cdots\cdots\cdots(\text{Ⅶ})$$

이 된다. 이들에서

$$\left.\begin{aligned}\sin C\sin A\cos b &= (-\cos C)(-\cos A)-(-\cos B) \\ \sin A\sin B\cos c &= (-\cos A)(-\cos B)-(-\cos C) \\ \sin B\sin C\cos a &= (-\cos B)(-\cos C)-(-\cos A)\end{aligned}\right\}$$

이 成立하고 따라서 第 2 cosine 法則

$$\left.\begin{aligned}\cos B &= -\cos C\cos A+\sin C\sin A\cos b \\ \cos C &= -\cos A\cos B+\sin A\sin B\cos c \\ \cos A &= -\cos B\cos C+\sin B\sin C\cos a\end{aligned}\right\}$$

가 誘導된다. 이 求法은 第 1 cosine 法則에 双對의 理를 適用한 方法이다.

다음에는 第 3 cosine 法則을 誘導하는 關係式(Ⅶ)에서 $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ 대신에 $\dfrac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}$, $\dfrac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}$, $\dfrac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}$를 代入하면

$$\left.\begin{array}{l}
\left(\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\right)\left(\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\times\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\times\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\right)+\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\times\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\cdot\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\times\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\\
\quad=\left(\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\right)\left\{\left(\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\right)\left(\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\cdot\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\right)-\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\right\}\\
\qquad+\left(\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\right)\left(\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\cdot\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\right)-\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\cdot\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
\left(\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\cdot\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\right)\left(\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\times\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\times\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\right)+\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\times\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\times\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\\
\quad=\left(\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\cdot\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\right)\left\{\left(\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\right)\left(\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\right)-\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\right\}\\
\qquad+\left(\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\cdot\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\right)\left(\frac{\boldsymbol{b}\times\boldsymbol{c}}{\sin a}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}\right)-\frac{\boldsymbol{a}\times\boldsymbol{b}}{\sin c}\cdot\frac{\boldsymbol{c}\times\boldsymbol{a}}{\sin b}
\end{array}\right\}(\mathrm{IX})$$

이 되고 여기서

$$\left.\begin{array}{l}
-\cos C(\sin C\sin A\cos b)+\sin C\sin B\cos a\\
\quad=-\cos C\{(-\cos C)(-\cos A)-(-\cos B)\}\\
\qquad+(-\cos B)(-\cos C)-(-\cos A)\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
-\cos B(\sin B\sin A\cos c)+\sin B\sin C\cos a\\
\quad=-\cos B\{(-\cos A)(-\cos B)-(-\cos C)\}\\
\qquad+(-\cos B)(-\cos C)-(-\cos A)
\end{array}\right\}$$

따라서

$$\left.\begin{array}{l}
-\cos C\sin C\sin A\cos b+\sin C\sin B\cos a\\
\quad=\cos A(1-\cos^2 C)=\cos A\sin^2 C\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
-\cos B\sin B\sin A\cos c+\sin B\sin C\cos a\\
\quad=\cos A(1-\cos^2 B)=\cos A\sin^2 B
\end{array}\right\}$$

이들에서 旣存의 公式

$$\left.\begin{array}{l}
\cos A\sin C=\sin B\cos a-\cos C\sin A\cos b\\
\cos C\sin B=\sin A\cos c-\cos B\sin C\cos a\\
\cos B\sin C=\sin A\cos b-\cos C\sin B\cos a\\
\cos C\sin A=\sin B\cos c-\cos A\sin C\cos b\\
\cos B\sin A=\sin C\cos b-\cos A\sin B\cos c\\
\mathrm{ocs}\,A\sin B=\sin C\cos a-\cos B\sin A\cos c
\end{array}\right\}$$

이 誘導된다. 단 (IX)의 첫째와 마지막式은 위의 公式의 첫째와 마지막式에 해당한다.

本稿는 일단 여기에서 멈춘다. 앞으로도 이것을 繼續하여 다시 發表할 機會가 있기를 바란다.