非線形 縮小半群에 關한 研究

劉 成 龍

A Note on Nonlinear Contraction Semi-groups

Sung Ryong Yoo

〈Contents〉

1. Introduction

Many authors have studied on Nonlinear semigroups. In 1969 Isao Miyadera proved; 〈Theorem A〉 Let $\{T(\xi);\ 0\leq\xi<\infty\}$ be a nonlinear contraction semigroup. Suppose that there exists a set D such that $D\subset D(A)$ and for any $x\in D$, $T(\xi)x\in D(A)$ for $a.e.\ \xi>0$.

Then convergence is uniform with respect so ξ in every compact subset of $[0,\infty)$ Hence if D is dense in X, then above fact holds good for all $x\in X$.

The first aim of this paper is to show the following theorem 1, 2 and theorem 3 in section 3 preparing to serve the above theorem A.

The second purpose of this paper is to verify the lemmas of which we take advantage of in order to show the theorem 1, 2 and theorem 3.

要 約

1969年에 Isao Miyadera 가 Banach 空間에서 非線形縮小半群에 關한 「定理 A」를 證明하였다.
本論文에서는 그 「定理 A」에 앞서 非線形縮小半群 $\{T(\xi;\ A_0);\ 0\leq\xi<\infty\}$에 關한 定理 1, 2 를 몇가지 基本的인 補助定理를 利用하여 밝혀 보였다.

2 Definition, propositons and Lemmas.

Let X_0 be a Banach space and Let $\{T(t);\ t\geqq0\}$ be a family of nonlinear operations X_0 into X_0 satisfying the following conditions.

1. $T(0)=I$ (the identity) $i.e.$, $T(0)x=x,\ x\in X_0$
2. $T(t+s)=T(t)\cdot T(s)$ for $t,s\geq0$
3. $\|T(t)x-T(t)y\|\leqq\|x-y\|$ for $x,y\in X_0$

4. $\lim_{t\to o+} \|T(t)x-x\|=0$ for $x\in X_o$

Such a family $\{T(t);\ t\geq 0\}$ is called a nonlinear contraction semigroup.

Now then the definition yields directly the following.

〈Proposition 1〉 From the above definition 4., for any fixed $x\in X_o$, $T(t)x$ is strongly continuous in $t\geqq 0$.

Proof: Put $t_2>t_1$, and $t_2-t_1=\tau$, we have

$\|T(t_2)x-T(t_1)x\|=\|T(t_1+\tau)x-T(t_1)x\|$

$=\|T(t_1)T(\tau)x-T(t_1)x\|\leqq\|T(\tau)x-x\|\to 0(\tau=t_2-t_1\to 0).$ □

And then we define the strong infinitetesimal generator A_o of $\{T(t);\ t\geqq 0\}$ on X_o by

$A_o x=\lim_{h\to o+} A_h x$ for all $x\in D(A_o)$, where $D(A_o)=\{x\in X_o;\ \exists \lim_{t\to o+} A_h x\}$, $A_h=h^{-1}(T(h)-I)$ and define the weak infinitetesimal generator A' of $\{T(t);\ t\geqq 0\}$ on X_o by

$A'x=\mathrm{w}-\lim_{h\to o+} A_h x$, $x\in D(A')$ where $D(A')=\{x\in X_o;\ \exists\ \mathrm{w}-\lim_{h\to o+} A_h x\}$

〈Proposition 2〉 Let $\{T(t);\ t\geqq 0\}$ be nonlinear contraction semigroup on X_o and $\lim_{t\to 0+}\frac{\|T(t)x-x\|}{t}=L<\infty$, $x\in X_o$

Then (1) $\|T(t+h)x-T(t)x\|\leq hL(t,\ h\geqq 0)$

(2) $\varphi(t)=\lim_{h\to 0} h^{-1}\|T(t+h)x-T(t)x\|$ exists and $\varphi(t)<\infty$ where $\varphi(t)$ is a monotonous decreasing function from $[0,\infty)$ into $[0,\infty)$.

proof; (1) For arbitrary $\varepsilon>0$, there exists $\{h_k\}$ such that $\|T(h_k)x-x\|<(L+\varepsilon)h_k$, $h_k>0$, $h_k\to 0$, for all $r_i\geqq 0(i=1,2,\cdots)$, $\tau\geqq 0$, we gain

$$\|T(\tau+\sum_{i=1}^{n}r_i)x-x\|=\|T(\sum_{i-1}^{n}r_i)T(\tau)x-T(\tau)x+T(\tau)x$$

$$-x|\leqq\|T(\sum_{i=1}^{n}r_i)x-x\|+\|T(\tau)x-x\|$$

$$=\|T(\sum_{i=1}^{n}r_i)x-T(r_1)x+T(r_1)x-x\|+\|T(\tau)x-x\|$$

$$\leqq\|T(\sum_{i=1}^{n}r_i)x-T(r_1)x\|+\|T(r_1)x-x\|+\|T(\tau)-x\|$$

$$\leqq\|T(\tau)x-x\|+\sum_{i=1}^{n}\|T(r_i)x-x\|,$$ since $t,h\geqq 0$,

$$\|T(t+h)x-T(t)x\|=\|T(t)T(h)x-T(t)x\|\leqq\|T(h)x-x\|$$

Since $h>0$, $n_k=\frac{h}{h_k}\left(n_k\leqq\frac{h}{h_k}<n_k+1\right)$ and so $\tau=h-n_kh_k\geqq 0$, $r_i=h_k(i=1,2,\cdots,n_k)$.

Hence $\|T(h)x-x\|=\|T(\tau+\sum_{i=1}^{n_k}h_k)x-x\|$

$$\leqq\|T(\tau)x-x\|+n_k\|T(h_k)x-x\|=\|T(h-n_kh_k)x-x\|+$$

$$+n_k\|T(h_k)x-x\|\leqq\|T(h-n_kh_k)x-x\|+(L+\varepsilon)h$$

$$k\to\infty ⇨ 0\leqq h-n_kh_k<h_k\to 0$$

Hence $\|T(h)x-x\|\leq(L+\varepsilon)h$, Thus $\|T(t+h)x-T(t)x\|\leq Lh$ □

(2) By The above assumption, since $\lim_{h\to o+}\sup\|T(h)x-x\|/h$

$\leq L$ and $L=\lim_{h\to o+}\inf\|T(h)x-x\|/h$, hence there exists $\lim_{h\to o+}\|T(h)x-x\|=L<\infty$.

Above all, for any $t>0$, $\|T(h)T(t)x-T(t)x\|\leqq$

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$\|T(h)x-x\|(h>0)$, $\liminf_{h\to o+} \|T(h)T(t)x-T(t)x\|/h \leq L < \infty$

and $\limsup_{h\to o+} \|T(h)T(t)x-T(t)x\| \leq L' = \liminf_{h\to o+} \|T(h)T(t)x-T(t)x\|/h < \infty$ and then using the assumption of (1) for the proof (1)

There exists $\lim_{h\to o+} \|T(t+h)x-T(t)x\|/h = L' < \infty$

On the hand, we take $t \geq s (t=s+a,\ a\geq 0)$. It becomes

$$\|T(t+h)x-T(t)x\| = \|T(a+s+h)x-T(s+a)x\| = \|T(a)T(s+h)x - T(a)T(s)x\| \leqq \|T(s+h)x-T(s)x\|$$

Hence $\lim_{h\to o+} \|T(t+h)x-T(t)x\| \leqq \lim_{h\to o+} \|T(s+h)x-T(s)x\|$

That is $\varphi(t) \leq \varphi(s)$□

From now we verify the following lemmas in order to prove theorms in section 4.

〈Lemma 1〉 $\{T(t);\ t\geqq 0\}$ is a nonlinear contraction semigrowp on X_o and

(1) For any $t\geqq 0$, $T(t)\hat{D} \subset \hat{D}$ where $T(t)\hat{D} = \{T(t)x;\ x\varepsilon\hat{D}\}$

(2) $X_o = X_o^{**}$(reflexive), because of X_o being Banach space from Definition, for any $x\varepsilon\hat{D}$, $T(t)x$ $\varepsilon D(A_o)$.

Then $\frac{d}{dt}T(t)x = A_oT(t)x$ (a, e, t)

Where $\hat{D} = \{x\varepsilon X_o;\ T(t)x:\ [0,\infty) \to X_o$, Lipschitz continuous on $D(f)\}$

Proof; For $t\geqq 0$, $x\varepsilon\hat{D}$

$$\|A_hT(t)x\| = h^{-1}\|T(t)T(h) - T(t)x\| \leqq h^{-1}\|T(h)x-x\| = \|A_hx\|(h>0)$$

Hence $\liminf_{h\to o+} \|A_hT(t)x\| \leq \liminf_{h\to o+} \|A_hx\| < \infty$ (since $x\varepsilon\hat{D}$)

Thus $T(t)x\varepsilon\hat{D}$. That is $T(t)\hat{D} \subset \hat{D}$□

Already we have found that f is a Lipschitz's continuous on $D(f)$ iff f continuous is on $D(f)$ and f satisfies Lipschitz's condition on $D(f)$.

〈Lemma 2〉 Let $x(t)$ be an X-valued function on an interval of real numbers Suppose $x(t)$ has a weak derivative $x'(t_o) \in X$ at $t=t_o$. If $\|x(t)\|$ is also differentiable at $t=t_o$.

Then $\|x(t_o)\|[d\|x(t)\|/dt]t=t_o = R_e(x'(t_o), f)$ for every $f\in F[x(t_o)]$

Proof; for all $f\in F(x(t_o))$, if $t>t_o$, $R_e[x(t)-x(t_o), f]$

$$= R_e(x(t), f] - \|x(t_o)\|^2 \leqq \|x(t)\|\|f\| - \|x(t_o)\|^2$$

$$= [\|x(t)-x(t_o)\|]\|x(t_o)\|$$

Thus $R_e[x'(t_o), f] \leq [d\|x(t)\|/dt]_{t=t_o}\|x(t_o)\|$ if $t<t_o$

On the hand,

$$R_e(x'(t_o), f) \geq [d\|x(t)\|/dt]_{t=t_o}\|x(t_o)\|$$

Hence $R_e(x'(t_o), f) = \|x(t_o)\|[d\|x(t)\|/dt]_{t=t_o}$□

Since $T(h)$ is a contraction and $A_h = h^{-1}[T(h)-I]$, Especially we heve, for all $x, y\in X_o$,

$$\|A_hx - A_hy\| = \frac{1}{h}\|T(h)x - T(h)y - (x-h)\| \leq \frac{1}{h}[\|T(h)x - T(h)y\| + \|x-y\|] \leq \frac{2}{h}\|x-y\|$$

3. Theorems and corollary

[Theorem 1] Let $T(t;\ A_o)$ be a nonlinecar contraction semigroup satisfying the following conditions:

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For each $x \in X_o$, $T(t;\ A_h)x$ is strongly continuously differentiable in $t \geqq 0$ then $\|(Tt:\ A_h)x - T(t:\ A_h)y\| \leqq \|x-y\|$ for all $x, y \in X_o$, $t \geqq 0$

Proof; Put $T(t:\ A_h)x = u(t:\ x)$ for $t \geqq 0$ and $x \in X_o$ since $u(t:\ x)$ is the unique solution of cordlary and $\{T(t;\ A_h):\ t \geqq 0\}$ is a contraction semigroup. Fix $x, y \in X_o$ and put $z(t) = u(t:\ x) - u(t;\ y)$. Cleary $z(t) \in C^1(([0,\infty):\ X)$ and $dz(t)/dt = A_h u(t:\ x) - A_h u(t:\ y)$, $z(0) = x - y$.

Since $\|z(t)\|$ is absolutely continuous, $\|z(t)\|$ is differentiable for $a, c, t \geqq 0$.

Therefore, by lemma 2, we get for $a.e.\ t_o \geqq 0$

$$\|z(t_o)\|[d\|z(t_o)\|/dt_o] = R_e(z'(t_o), f(t_o))$$

$$= R_e[A_h u(t_o:\ x) - A_h u(t_o:\ y),\ f_{t_o}] \leqq 0 \text{ for every } f_{t_o} \in F(z(y))$$

$$= F(u(t_o:\ x) - u(t_o:\ y))$$

Hence $\|z(t_o)\|[d\|z(t_o)\|/dt_o] \leqq 0$ for $a, e, t_o \geqq 0$

And since $\|z(t)\|^2 - \|z(o)\|^2 = \int_o^t [d\|z(t_o)\|^2/dt_o]dt_o = 2\int_o^t [\|z(t_o)\| \cdot d\|z(t_o)\|/dt_o]dt_o \leqq 0$

That is, for each $t > 0$,

$$\|z(t)\|^2 - \|z(0)\|^2 = \|u(t:\ x) - u(t:\ y)\|^2 - \|x-y\|^2 \leqq 0$$

We obtain

$$\|T(t:\ A_h)x - T(t:\ A_h)y\| \leqq \|x-y\| \square$$

[Theorem 2] Let $\{T(t):\ 0 \leq t\}$ be a nonlinear contraction semigroup. For each $\delta > 0$, there exists a nonlinear contraction semigroup $T(t:\ A_h);\ 0 \leq t\}$ with the infinitesimal generator $A_h (= h^{-1}[T(h) - I])$ satisfying the same condition, of theorem 1.

Then $dT(t:\ A_h)x/dt = A_h T(t;\ A_h)x$ for $t \geqq 0$

Proof; Using Lemma 1, Since $x \in \hat{D}$, $T(t;\ A_h)$ is Lipschitz continuons with respect to t. Thus we gain

$$\frac{d}{dt} T(t:\ A_h)x = \lim_{h \to o+} \frac{1}{h}\Big[T(t_o + h:\ A_h)x - T(t_o:\ A_h)x\Big]$$

$$= \frac{1}{h}(T(t_o+h)x - T(t_o)x) = \frac{1}{h}(T(h_o)T(t_o)x - T(t_o)x)$$

$$= T(t_o)x \cdot h^{-1}(T(h) - I) = A_h \lim_{h \to o+} T(t_o + h)x$$

$$= A_h T(t_o)x = A_h \cdot T(t_o:\ A\delta)x \text{ (Because } T(t)x$$

$$= \lim_{h \to o+} T(t:\ A\delta)x)), \text{ for } x \in X_o,\ t \geqq 0. \square$$

In the above results, we find that it is possible to show [theorem 2] without using Lemma 1.

Now we can verify the above theorem 2, with making use of the following corollary 1. Let $u(t:\ x)$ be the unique solution in corollary 1 and put.

$T(t:\ A_h)x = u(t;\ x)$ for $t \geqq 0$, then $\{T(t:\ A_h);\ 0 \leqq t \leqq \infty\}$ is an nonlinear contraction semigroup satisfying the condition of [Theorem 1].

〈corollary 1〉 The equation $\dfrac{du(t:x)}{dt} = A_h u(t:\ x)$ for $t \geqq 0$, $u(0:\ x) = x$

has a unique solution $U(t:\ x) \in C^1([0,\infty):\ X)$ for any $x \in X$.

Where $C^1([0,\infty):\ X)$ denotes the set of all strongly continuonsly differentiable x-valued functions defined on $[0,\infty)$.

[corollary 2] $\{T(t:\ A_h);\ 0 \leq t\}$ is continuous in $t (\geqq 0)$ where $T(t:\ A_h) = \mathbf{exp}(tA_h)$.

proof; Since $T(t;\ A_h)x$ is a function from $[0,\infty)$ into X_o, Putting $\tau = T_2 - T_1$, we obtain the

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following

$$\|T(t_2;\ A_h)x-T(t_1;\ A_h)x=\|T(t_1+\tau;\ A_h)x-T(t_1;\ A_h)x\|$$
$$=T(t_1;\ A_h)T(\tau;\ A_h)x-T(t;\ A_h)x\|$$
$$\leqq\|T(\tau;\ A_h)x-x\|\to 0(\tau\to 0)\square$$

Owing to the above corollary and theorems, thus we have the following.

[Theorem 3] Under the assumption of [theorem 2] supposed "for each bounded set B in X_0 and $t\geqq 0$.

Then $\sup_{x\in B}\|(Tt+\delta;\ A_h)x-T(t;\ A_h)x\|\to 0$ as $\delta\to 0$."

4. Conclusion

With the aid of the above Lemmas, corollaries and theorems, we can verify [theorem A] in Introduction and find that J, R Dorroh[2] has obtained the same result of [theorem A] under the assumption "for each $x\in D$, $T(\xi)x$ is strongly conitnuously differentiable in $\xi\geqq 0$."

Thus we find that Y, kōmura[3] has proved that if X is a Hilbert space, then for each $x\in D(A)$

$$T(\xi)x\in D(A)\ \text{for}\ a,e,\xi\geqq 0,\ \text{and}\ T(\xi)x=x+\int_0^\xi AT(\gamma)xd\gamma.$$

Therefore we have the following without the above assumption, the convergence of I. Miyadera [1] holds for each $x\in D(A)$.$\square$

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