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A STUDY ON STOCHASTIC MODEL OF THE RENEWAL PROCESS

(Under Non-Markovian of Continuous Time
with Condition Space of Discrete Type)

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ABSTRACT: The purpose of this paper is to give an economical help when purchasing a car, by making a model of the correlation between the price of a car and the maintenace cost that arises from an impact, and by giving various numerical example.

1. Introduction

It has been considered that almost all of the life span of an individual depend upon exponential distribution. In this paper, an individual life span cannot alway be regarded as exponential random variables. And as to a stochastic process model of each individual number, taking into consideration the present age to an individual (that is, how long the individual has been used since it was made up), the future individual number only depends not only upon the present individual number but also upon the age in which each individual is being situated. As a result, supposing the individual number and each individual's age as a set, this becomes Markovian. therefore, in this paper, the individual number is treated as an invariable, without depending upon the whole record of each individual since its birth.

To begin with let's examine, as an example, a case of a renewal process. An electrical bulb has an ability to light. When the bulb is not working, it is replaced with a new one. In this case, the life span of the new one is indepe-

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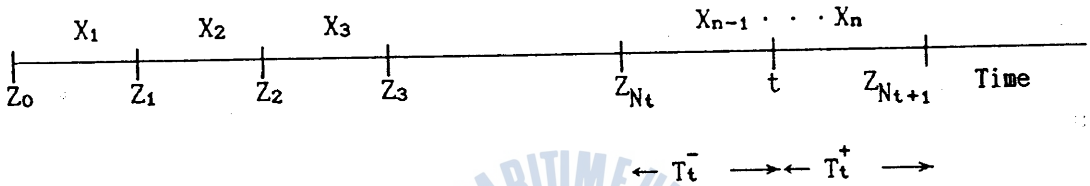
ndent random variables X_t and suppose all of them were of the same distribution as a positive random variable X, the first individual would be established at time 0.

The time Z_n until the N^{th} individual goes out of order can be regarded as the random walk

$$Z_n = X_1 + X_2 \cdots\cdots\cdots + X_n,\ X_0 = 0$$

Now, let us fix the time t, and observe each porbability random variable as follows,



A) Recurrence (Regeneration) number between(0,t)

$N_t = \mathrm{Max}\{n : Z_n \leq t\}$

B) Forward recurrence time (Excess life)

$T_t^+ = Z_{N_{t+1}} - t$

C) Backward recurrence time (current life)

$T_t^- = t - Z_{N_t}$

Here, when the T is fixed,

$T_t^- + T_t^+ = Z_{N_{t+1}} - Z_{N_t} = X_{N_{t+1}}$

does not follow the same distribution as X. It is because, as to the performance value of the process, the fixed time t more easily enters the longer duration than shorter time interval.

For this reason, as T^+ depends on T^-, this process does not become markovian. We have for example, two kinds of bulb, of which one lasts only one week, and the other lasts one year.

If the other one has lasted for six months, it is guaranteed that it will last another six months. Therefore, the stochastic process model $N_t ; t \leftarrow (1, \infty)$ that represents the number of individuals is the mapping random walk $\{Z_n ; n=0,$

1,2 · · · · · · }.

That is, the fact that the recurrence frequency until the time t is at least n, and that the time required for the number of individuals to recur until N^{th} should not exceed t are the same mapping.

consequently, we have

$$P_r(N_t \geq n) = P_t(Z_n \leq t).$$

The main purpose of this paper, by obtaining an integral equation for the renewal function and renewal density, is to make a purchasing model available by means of the renewal process.

2. Integral equation for the renewal function and renewal density.

Since Z_{N_t} is the time of the last renewal prior to or at time t and Z_{N_t+1} is the time of We have that

$$Z_{N_t} \leq t \leq Z_{N_t+1}$$

$$\text{or } \frac{Z_{N_t}}{N_t} \leq \frac{t}{N_t} < \frac{Z_{N_t+1}}{N_t}$$

However, since $\frac{Z_{N_t}}{N_t} = \sum_{i=1}^{N_t} \frac{Z_i}{N_t}$ is the average of N_t indepedent and identity distributed random variables, it follows by the strong law of large numbers that

$$\frac{Z_{N_t}}{N_t} \longrightarrow \mu \text{ as } N_t \longrightarrow \infty.$$

But, since $N_t \longrightarrow \infty$ when $t \longrightarrow \infty$, we obtain $\frac{Z_{N_t}}{N_t} \longrightarrow \infty$ as $t \longrightarrow \infty$.

Furthermore, writing $\frac{Z_{N_t+1}}{N_t} = \left(\frac{Z_{N_t+1}}{N_t+1}\right)\left(\frac{N_t+1}{N_t}\right)$ we have that $\frac{Z_{N_t+1}}{N_t+1} \longrightarrow \mu$ by the same reasoning as above and $\frac{Z_{N_t+1}}{N_t} \longrightarrow 1$ as $t \longrightarrow \infty$.

Hence, $\frac{Z_{N_t+1}}{N_t} \longrightarrow \mu$ as $t \longrightarrow \infty$.

Therefore, $\frac{t}{N_t} \longrightarrow \mu \ (t \longrightarrow \infty)$.

On condition that density functions $f(x)$ $(0 < x < \infty)$ is continuous random variable, we have the lemma as follow.

LEMMA p g f$\Pi_t(s)$ of N is

$$\Pi_t(s) = s\int_0^t f(u)\Pi_{t-u}(s)du + \int_t^\infty f(u)du$$

PROOF the partition by the first regeneration time u is used

$$\text{If } u > t,\ N_t = 0$$

$$\text{If } u < t,\ N_t = N'_{t-u}$$

Where, the N'_{t-u} follows regeneration number of times N_{t-u} and the distribution from the remaining t-u to t.

At a point of times regeneration, can be regarded as a new one. Let us make use of Markovian at a point of times.

$$\text{As } \Pr\{U \in (u, u+du)\} = f(u)du$$

$$E[S^{N_t}]$$

$$= E_u[E_{N_t} | U = u\ S^{N_t}]$$

$$= \int_0^\infty f(u)du[E_{N_t} | U = u\ S^{N_t}]$$

$$= \int_0^t f(u)du\ E_s 1 + N_{t-u} + \int_t^\infty f(u)duE_{s0}$$

That is,

$$(1)\ \Pi_t(s) = s\int_0^t f(u)\Pi_{t-u}(s)du + \int_t^\infty f(u)du$$

(Q.E.D)

Here,

If the above formula (1) is used, the following the renewal function (mean value)

$$H_t = E(N_t)$$

$$= \sum_{n=1}^{\infty} P\{N_t \geq n\}$$

$$= \sum_{n=1}^{\infty} P\{S_n \leq t\}$$

and

the renewal density

$$h_t = \frac{d}{dt} H_t$$

obtain Model 1.

MODEL1 The renewal function(the mean value) is

$$H_t = E(N_t)$$
$$= \int_0^t f(u)du + \int_0^t f(u)H_{t-u}du,$$

and the renewal density is

$$h_t = \frac{d}{dt}(H_t)$$
$$= f(t) + \int_0^t f(u)h_{t-u}du.$$

PROOF From(1)

$$\frac{\partial}{\partial s}\Pi_t(s) = \int_0^t f(u)\Pi_{t-u}(s)du + s\int_0^t f(u)\frac{\partial}{\partial s}\Pi_{t-u}(s)du$$

Hence,

$$H_t = \int_0^1 f(u)du + \int_0^t f(u)H_{t-u}du \quad (\because s \to 1)$$

If the above equation is differentiated for t,

$$\frac{\partial}{\partial t}H_t = \frac{\partial}{\partial t}\int_0^t f(u)du + \frac{\partial}{\partial s}\int_0^t f(u)H_{t-u}du$$

Therefore,

$$(2)\quad h_t = f(t) + H_0 + \int_0^t f(u)h_{t-u}du$$

$$= f(t) + \int_0^t f(u)h_{t-u}du$$

$(\because H_0 = E[N_0] = E[o] = 0)$ (Q.E.D)

Here,

$$N_t - \frac{t}{\mu}\ (t \to \infty)\ (\because n^{-1}Z_n \to \mu)$$

therefore,

$$H_t - \frac{t}{\mu}$$

and

$$h_t - \frac{1}{\mu}$$

From (2)

$$h_t = f(t) + \int_0^t f(u)h_{t-u}du \text{ is changed to}$$

$$h_t\Delta + O(\Delta) = \{f(t)\Delta + O(\Delta)\} + \int_0^t \{f(u)du\}.$$

$$\{h_{t-u} + O(\Delta)\}$$

The above equation happens within

P_r {the first regeneration happens within (t, t+Δ), but any u (0, t) becomes (u, u+du), so the first regeneration happens, and there is another regeneration happens, and there is another regeneration within the time as long as t-u

That is,

when regeneration happens at time 0, as there a probability that regeneration will happen between (t,t+Δ).

Therefore,

$$h_t = P_r\{\text{a renewal happens within } (t,\ t+\Delta) \mid \text{renewal happen at } 0\}$$

$$= h_t\Delta + O(\Delta)$$

Here,

The following MODEL2 can be obtained.

MODEL2 The distribution to forward recurrence time T_t^+ of renewal process is

$$g_t^+(x) = f(t+x) + \int_0^t h_{t-u}\ f(u+x)\ du \text{ it's limit distribution being}$$

$$g(x) = \int_0^\infty \frac{1}{\mu} f(u+x)dn = \frac{1}{\mu}\int_x^\infty f(\omega)d\omega = \frac{1}{\mu}\{1-F(x)\}$$

PROOF The us obtain density g_t^+ of T_t^+ with disjointing by the last renewal before t.

$P_r\{T_t^+ \in (t, t+\Delta)\}$ = P_r\{the first renewal may happen either aocording to $(t+x, t+x+\Delta)$ $(x>0)$, while the last renewal before t may happen it $(v, v+dv)$ for $v\in(0,t)$, and next renewal may happen at $(t+x, t+x+\Delta)$, that is, at late as $(t+x-v, t+x-v+\Delta)$

Therefore,

$$g_t^+(x) + O(\Delta)$$

$$=\{f(t+x)\Delta + O(\Delta)\} + \int_0^t h_v dv\, f(t+x-v) + O(\Delta)\}.$$

Hence,

put $V = t-v$, divided by Δ

$$= g_t^+(x) = f(t+x) + \int_0^t h_{t-u}\, f(u+x)$$

Suppose, $f(x) \longrightarrow 0(x-\infty)$, the above equation is formed as to fixed t, but when $t \longrightarrow \infty$, h_{t-u} is obtained μ^{-1}.

That is,

$$g_t^+(x) \rightarrow g(x) = \int_0^\infty \frac{1}{\mu} f(u+x)dx$$

$$= \frac{1}{\mu}\int_x^\infty f(u)du$$

$$= \frac{1}{\mu}\{1-F(x)\}.$$

From MODEL2, the following MODEL3 can be obtained.

MODEL3 The distribution to backward recurrence time T_t^- of renewal process is

$g_t^-(x) = h_{t-u}\{1-F(x)\}$, $(0<x<t)$,

it's limit distribution approaches the density that has density

$$g(x) = \frac{1}{\mu}\{1-F(x)\}, \quad (0<x<t).$$

PROOF, $P_r\{T_t^- \in (x, x+\Delta)\}$

= P_r {a renewal does't happen at $(0,t)$ $(T_t^-=t)$, but a renewal happens at $(t-x, t-x+\Delta)(0<x<t)$, and no renewal at $(t-x, t)$\}

Accordingly,

P_r {a renewal doesn't happen to (u,r)| a renewal happens at u}.

Therefore,

$P_r(T_{\bar{t}}=t) = P_r(x > t) = 1-F(t)$

$g_{\bar{t}}(x) = h_{t-x}\{1-F(x)\}$ (o x t)

That is,

the distribution of $T_{\bar{t}}$ has a probability at a point t, and density $g_{\bar{t}}(x)$ at (o,t).

Again,

when $t \longrightarrow \infty$, $F(t) \longrightarrow 1$, $h_{t-x} \longrightarrow \frac{1}{\mu}$

Accordingly,

the distribution of $T_{\bar{t}}$ approaches the density

$g(x) = \{1-F(x)\}/\mu$ $(0 < x < t)$ (this is the same as the limit of $T_{\bar{t}}$).

3. A car purchasing Model

The life span of a car with a life-cycle is a continuous stochastic variable with any distribution and density. Now, suppose professor Kim had had his car broken down or had planned to purchase a new car in T year, and the price of the new car were P_1, and the would pay P_2 every time his car breaks down. If purchasing a car, he uses it until it is scrapped. How long should he use it and purchase a car most economically?

Solution: Suppose a life-cycle is the time from his purchase of the old can to the purchase of a new one when it gets entirely unusable, the average cost required for the maintenance of it is

E[the whole sum of money required for a life-cycle]

Now, if X is the life span of professor Kim's can during an arbitrary length of time, the cost required for the car is

P_1 (when $x > T$)

or

$$P_1 + P_2 \ (\text{when } x \le T).$$

Therefore,

the average cost for a car during a life-cycle is

$$P_1 \cdot P_r\{x > T\} + (P_1+P_2)\ P\{x \le T\} = P_1 + P_2\ H_t(T)$$

Where, $H_t(T)$: renewal function during T.

And,

the length of life-cycle is

$x \le T \longrightarrow X$

$x > T \longrightarrow T$

Here,

the average length of life-cycle from Lemma, MODEL1, MODEL2,

$$\int_0^T x h_t(x)dx + \int_T^\infty T h_{t-u}(x)dx$$

$$= \int_0^T x h_t(x)dx + T(1-H_t(T))$$

is obtained.

Therefore,

the average life span for the price is

$$\frac{P_1 + P_2\ H_t(T)}{\int_0^T x h_t(x)dt + T(1-H_t(T))}$$

4. Numerical Examples

Table 1: Uniformly distribution (Unit:₩ ten thousand)

Interval / Price	(0,10)	(0,11)	(0,12)	(0,13)	(0,14)	(0,15)
450	9.5	10.4	11.2	12.0	12.8	13.1
500	9.5	10.4	11.3	12.1	12.9	13.2
600	9.6	10.5	11.4	12.2	13.1	13.9

Table 2: Gamma distribution (Unit:₩ ten thousand)

Interval Price	(0,10)	(0,11)	(0,12)	(0,13)	(0,14)	(0,15)
450	10.02	11.01	12.00	13.00	14.00	15.00
500	10.03	11.01	12.00	13.00	14.00	15.00
600	10.03	11.01	12.01	13.00	14.00	15.00
1000	10.05	10.02	12.01	13.00	14.00	15.00

Where, $P_{10}=50$, $P_{11}=60$, $P_{12}=70$, $P_{13}=80$, $P_{14}=90$, $P_{15}=100$.

As seen on the above distribution Table1, Table2, it is thought that the most suitable life-cycle of a car purchasing model is uniformly distribution and about 10 years is most economical. In reality, as this distribution the replacement effect (of a new can with an old one) is most economical when done in about 4 years.

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